Stability of Planar and Non-Planar Libration Points in the Photo-gravitational ER4BP with Oblate Primaries

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Abstract. This paper studies the location and stability of planar (collinear) and non-planar (out of plane) libration points in the elliptic restricted four body problem under the influence of radiation pressure and oblateness effects. We have assumed that three bodies are moving in elliptical orbit around their common center of masses fixed at origin of the coordinate system and they are always at the vertices of an equilateral triangle while the fourth body is infinitesimal mass. It was observed that for this system, two collinear and two or four out of plane equilibrium points exist which is depend on true anomaly of the orbit. Out of plane equilibrium points lie on the $x - z$ plane in symmetrical position with respect to x-y plane. Both Planar (collinear) and non-planar (out of plane) points are found unstable in linear sense. The fractal Basin for planar (collinear) and non-planar (out of plane) libration points were studied and found to depend on radiation pressure.
Key words: Elliptic restricted four body problem, Oblateness, Radiation pressure, Fractal Basin

Introduction

In the present scenario, the Four Body Problem is an important extension of the very well studied three body problem. R4BP is often more realistic for certain applications such as space flight mission or satellite positioning then the three body problem. For instance, while positioning the satellite orbiting Moon, the consideration of the gravitational forces of Earth and Moon is not sufficient, the effect due to Sun is also a great perturbing force affecting the motion of satellite. Including different perturbing forces, several author studied the Circular restricted four body problem (CR4BP). In CR4BP the primaries move in an circular orbit and in elliptic restricted four body problems, the primaries move in an elliptical Keplerian orbit. Most of the celestial bodies move in an elliptical orbit. So the consideration of the non-zero value of the eccentricity of orbits of primaries is more equipped to analyse the dynamical system more accurately.

Grebenikov [7] presented the Newtonian circular restricted four-body problem and obtained nonlinear algebraic equations determining equilibrium solutions in the rotating frame and found six possible equilibrium configurations of the system. El-Saftawy and Abd El-Salam [9] defined restricted four body problem, in the bicircular model. By constructing the Hamiltonian of the problem, they obtained solution using Delva Hanslmeir perturbation technique. Shoaib and Ibrahima [10] discussed the equilibrium solutions of four different types of collinear four-body problems having two pairs of equal masses. Baltagiannis and Papadakis [5] have investigated the stability of the libration points in the restricted four body problem. They have shown that ten libration points exist and out of which two or four are collinear and the remaining are non-collinear. The photogravitational restricted four body problem is the classical problem if at least one of the interacting bodies is an intense emitter of radiation. The effect of radiation pressure of a source on a small particle, is expressed by means of the reduction factor q. Papadouris and Papadakis [16]have studied the photogravitational restricted four body problem. Singh and Vincent [8] studied the out of plane equilibrium points. They considered all the primaries as radiating bodies and two of the primaries having the same radiation and mass value. They found that radiation pressure effects the location of the equilibrium points and the zero velocity curves. These points are found to be unstable. It has been observed that radiation pressure and oblateness of the primaries have remarkable influence on the existence and stability of equilibrium points in the restricted problems working on the approach.

Further Kumari and Kushvah [11] investigated equilibrium points under the oblateness effects of first two bigger primary. They established eight equilibrium points, two collinear equilibrium points and six non-collinear equilibrium points and observed that the stability regions of the equilibrium points expanded due to the presence of oblateness coefficients for various value of Jacobi constant C and also shown that the fractal basins of attraction for the equilibrium points.

The above discussion of the major work undertaken by scientists shows that circular restricted four body problem has been studied extensively, though scope for further development still remains.

However study of elliptic R4BP based on Langrang's solution has not been undertaken by many authors. In the elliptic restricted four body problem the three primaries are moving in elliptic orbits and the problem is restricted in the sense that the motion of the three primaries as well as the infinitesimal mass take place in a plane called the plane of motion. The fourth body of infinitesimal mass does not affect the dynamics of the problem.

Assadian et al [1] studied the effect of the Sun on the Lagrange points of the Earth Moon system in the frame of bielliptic restricted four body problem (BiERFBP) in which the motion of Earth around the Sun is presumed to be elliptic orbit in the ecliptic plane. Also the motion of Moon around the Earth is presumed to be elliptic but out of the ecliptic plane.

Chakraborty and Narayan [3] studied the bielliptic restricted four body problem. On the other hand Chakraborty and Narayan [4] discussed the equilibrium points, their stability, the pulsating ZVC and fractal basin for the elliptic triangular restricted four-body problem.

Extending this work, we have modeled the problem according to solution of Lagrange, where they are always at the vertices of the equilateral triangle, while the fourth body is infinitesimal. We have shown the existence of the non-collinear points and determined their locations numerically in the elliptic Restricted four body problem with radiation pressure and oblateness. [6]. In this paper we have undertaken the study of the position and stability of planar (collinear) and non-planar (out of plane) libration points. The fractal basins of the model are also explored. MATHEMATICA 10 software was employed for graphical and numerical solutions in this paper. The rest of paper is organized as follows: Section 1 provides the equation of motion; Section 2 gives the position of the Planar (collinear) and Non-Planar (out of plane) equilibrium points; Section 3 focus on the stability of the Planar (collinear) and Non-Planar (out of plane) equilibrium points; In Section 4 the basin of attraction adopting the Newton Raphson Method has been discussed. The discussion and conclusion are drawn in Section 5.

1. Equations of motion

We have taken the configuration of the system as three bodies s_1, s_2 and s_3 of masses m_1, m_2 and m_3 where m_1 mass is greater than m_2 and m_2 equal to the m_3 moving in a plane about their center of mass O in Keplerian elliptical orbit having eccentricity e. This is further assumed that the bigger primary m_1 is radiating body and other two smaller primaries m_2 and m_3 are oblate spheroids. A third body P of infinitesimal mass is the mutual gravitational attraction of the three primaries but without affecting their motion. The motion of the infinitesimal is affected by the primaries. The oblateness parameter of the second and third primary are given by

$$A_1 = \frac{a^2}{5R^2}, A_2 = \frac{b^2}{5R^2}; \tag{1}$$

where $'a'$ and $'b'$ are the semi-major axis and R is the distance between the primaries. The radiation factor of the largest primary given by q is derived from the relation.

$$F = f_g - f_p = f_g(1 - \frac{f_p}{f_g}). = (1 - q)f_g \tag{2}$$

Here f_g is the gravitational attraction force, f_p is the radiation pressure force and q is the mass reduction factor The dimensionless variables are introduced by using the distance r between the primaries given by

$$r = \frac{a(1-e^2)}{1 + e\cos f};$$

where $'a'$ and $'e'$ are the semi-major axis and the eccentricity of the elliptical orbit of the either primary around the other and f is the true anomaly. The angular motion of infinitesimal moving in elliptical orbit based on Kepler's laws is given by

$$\frac{df}{dt} = \frac{na^2\sqrt{(1-e^2)}}{r^2}; \tag{3}$$

We get the equation of motion of the infinitesimal in non-dimensional barycentric, pulsating and non-uniformly rotating coordinate system $(\overline{x}, \overline{y}, \overline{z})$ written in the form [6]:

$$\overline{x}'' - 2\overline{y}' = \frac{\partial \Omega}{\partial \overline{x}};$$

$$\overline{y}'' + 2\overline{x}' = \frac{\partial \Omega}{\partial \overline{y}}; \tag{4}$$

$$\overline{z}'' = \frac{\partial \Omega}{\partial \overline{z}};$$

where,

$$\Omega = \frac{1}{1+ecosf}\omega; \tag{5}$$

$$\omega = (\frac{\overline{x}^2+\overline{y}^2-\overline{z}^2ecosf}{2} + \frac{1}{n^2}\left[\frac{(1-2\mu)q}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} + \frac{\mu}{r_3} + \frac{\mu A_1}{2r_3^3}\right]). \tag{6}$$

$$\begin{aligned} r_1^2 =& (\overline{x}-\sqrt{3}\mu)^2+\overline{y}^2+\overline{z}^2; \\ r_2^2 =& (\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2})^2+(\overline{y}-\frac{1}{2})^2+\overline{z}^2; \\ r_3^2 =& (\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2})^2+(\overline{y}+\frac{1}{2})^2+\overline{z}^2. \end{aligned} \tag{7}$$

Here the mean motion of the system is presented as follows

$$n^2 = \frac{(1+e^2)^{3/2}}{a^3(1-e^2)}\left[1+\frac{3A_1}{2}+\frac{3A_2}{2}\right] \tag{8}$$

Since these primaries are fixed in this coordinate system, the position are represented as $(\sqrt{3}\mu, 0, 0)$,$(-\frac{\sqrt{3}(1-2\mu)}{2}, -1/2, 0)$ and $(-\frac{\sqrt{3}(1-2\mu)}{2}, 1/2, 0)$ where $\mu = \frac{m_2}{m_1+m_2+m_3} = \frac{m_3}{m_1+m_2+m_3}$. From equation (4), we observe that

$$\begin{aligned} \frac{\partial\omega}{\partial\overline{x}} = \Bigg\{\overline{x} - \frac{1}{n^2}\Bigg\{ & \frac{(1-2\mu)(\overline{x}-\sqrt{3}\mu)q}{r_1^3} + \frac{3(\mu)(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2})A_1}{2r_3^5} \\ & + \frac{\mu(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2})}{r_2^3} + 3\frac{\mu(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2})A_2}{2r_2^5} \\ & + \frac{\mu(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2})}{r_3^3}\Bigg\}\Bigg\} \end{aligned}$$

$$\begin{aligned} \frac{\partial\omega}{\partial\overline{y}} = \Bigg\{\overline{y} - \frac{1}{n^2}\Bigg\{ & \frac{(1-2\mu)\overline{y}q}{r_1^3} + 3\frac{(\overline{y}+\frac{1}{2})A_1}{2r_3^5} + \frac{\mu(\overline{y}-\frac{1}{2})}{r_2^3} \\ & + 3\frac{\mu(\overline{y}-\frac{1}{2})A_2}{2r_2^5} + \frac{\mu(\overline{y}+\frac{1}{2})}{r_3^3}\Bigg\}\Bigg\} \end{aligned}$$

$$\frac{\partial \omega}{\partial \overline{z}} = -\overline{z}ecosf - \frac{1}{n^2}\Big\{\frac{(1-2\mu)\overline{z}q}{r_1^3} + 3\frac{\overline{z}A_1}{2r_3^5} + \frac{\mu\overline{z}}{r_2^3} + 3\frac{\mu\overline{z}A_2}{2r_2^5} + \frac{\mu(\overline{z})}{r_3^3}\Big\}\Big\} \tag{9}$$

2. Equilibrium points

2.1. Position of Planar (Collinear) Equilibrium Points

If the problem is reduced to planar form that is projection on the $\overline{xy-}$ plane considered, then the equilibrium points are obtained by solving the following equation: $\Omega_{\overline{x}} = 0$; $\Omega_{\overline{y}} = 0$.

Solving the above two equations by further imposing the condition $\overline{y} = 0$ we obtained collinear points by solving following equations :

$$r_1 = |(\overline{x} - \sqrt{3}\mu)|;$$

$$r_2 = r_3 = \sqrt{(\overline{x} + \frac{\sqrt{3}(1-2\mu)}{2})^2 + \frac{1}{4}};$$

$$\Big\{\overline{x} - \frac{1}{n^2}\Big\{\frac{(1-2\mu)(\overline{x}-\sqrt{3}\mu)q}{r_1^3} + \frac{3(\mu)(\overline{x} + \frac{\sqrt{3}(1-2\mu)}{2})A}{2r_2^5} + \frac{2\mu(\overline{x} + \frac{\sqrt{3}(1-2\mu)}{2})}{r_2^3}\Big\}\Big\} = 0$$

Also we note that when

$$r_2 = r_3$$

$A_1 = A_2 = A$ (say) In the present work, we have considered the small mass $\mu = 0.1$ and varying value of q and A_i, there exist collinear equilibrium points which are denoted by L_1, L_2. Fig. 1 represent the first and second equation of system of equation (9) with added condition $\overline{y} = 0$ and

$$r_2 = r_3.$$

They show the shift in the position of collinear equilibrium points for varying values of q and A.

2.2. Position of Non Planar (Out of Plane) Equilibrium Points

The position of the out of plane equilibrium points can be found from the equations of motion by setting all velocity and acceleration components equal to zero and solving the resulting system.

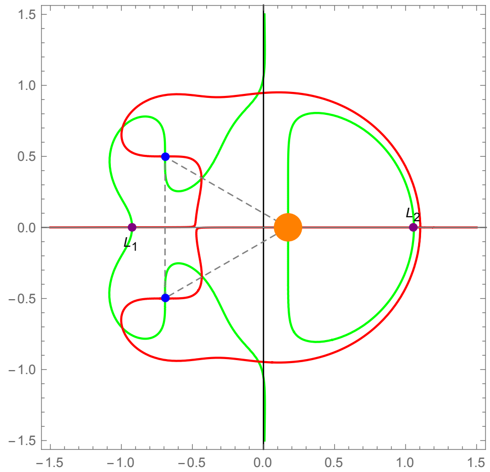

Fig. 1. Curve of Collinear Point for values of $\mu = 0.1$, $A = 0$ and $q = 1$.

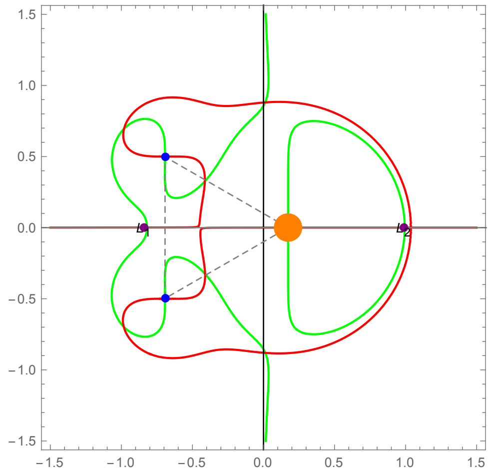

Fig. 2. Curve of Collinear Point for values of $\mu = 0.1$, $A = 0$ and $q = 0.9$.

When $y = 0$, $r_2 = r_3$ in the $x - z$ plane we have

$$
\begin{aligned}
r_1 =& \sqrt{(\overline{x} - \sqrt{3}\mu)^2 + \overline{z}^2};\\
r_2 =& r_3 = \sqrt{(\overline{x} + \frac{\sqrt{3}(1-2\mu)}{2})^2 + \frac{1}{4} + \overline{z}^2};
\end{aligned}
$$

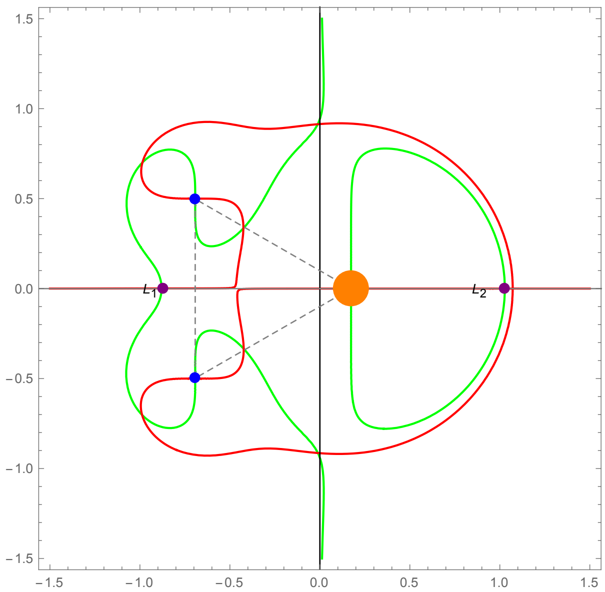

Fig. 3. Curve of Collinear Point for values of $\mu = 0.1$, $A = 0$ and $q = 0.8$.

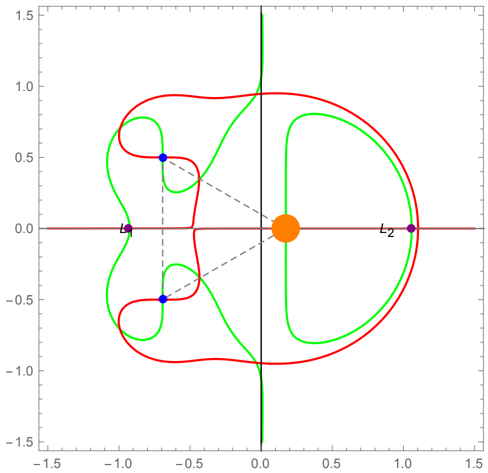

Fig. 4. Curve of Collinear Point for values of $\mu = 0.1$, $A = 0.0001$ and $q = 1$.

Then

$$\overline{x} - \frac{1}{n^2}\left\{ \frac{(1-2\mu)(\overline{x}-\sqrt{3}\mu)q}{r_1^3} + \frac{3\mu(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2})A}{2r_2^5} + \frac{2\mu(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2})}{r_2^3} \right\} = 0 \tag{10}$$

$$-\overline{z}e\cos f - \frac{1}{n^2}\left\{ \frac{(1-2\mu)\overline{z}q}{r_1^3} + 3\frac{\overline{z}A}{r_2^5} + 2\frac{\mu\overline{z}}{r_2^3} \right\} \Bigg\} = 0 \tag{11}$$

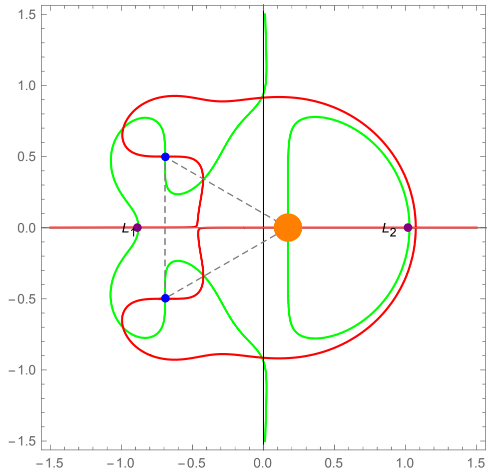

Fig. 5. Curve of Collinear Point for values of $\mu = 0.1$, $A = 0.0001$ and $q = 0.9$.

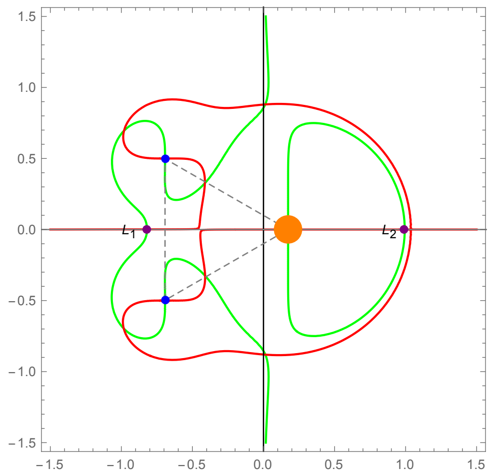

Fig. 6. Curve of Collinear Point for values of $\mu = 0.1$, $A = 0.0001$ and $q = 0.8$.

In the present work, we have considered the small mass $\mu = 0.1$ and varying value of q and A_i there exists non-planar equilibrium points which are denoted by L_{01}, L_{02}, L_{03}, L_{04}. Fig. $6-12$ represent the equation (10) and (11) in the $xz-$plane the black dot represents out of plane points. Their are position studied through numerical method. They are located in the (x, z) plane in symmetrical position with respect to the $x-$axis Fig. 8 and Fig. 9 show the shift in the position of out of plane equilibrium points for negative value of q for $f = 3\pi/2$ and $f = \pi$, where as Fig. 10 and Fig. 11 show the shift for positive value of q.

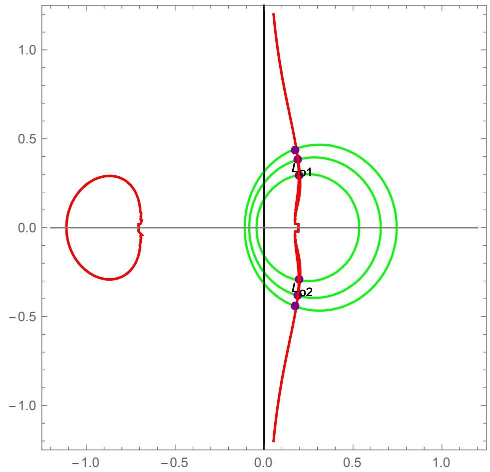

Fig. 7. The curves on $\bar{x}\bar{z}$ plane of $\mu = 0.1$, $A = 0.0001$ and $q = -0.01, -0.02, -0.03$ and $f = 2\pi$.

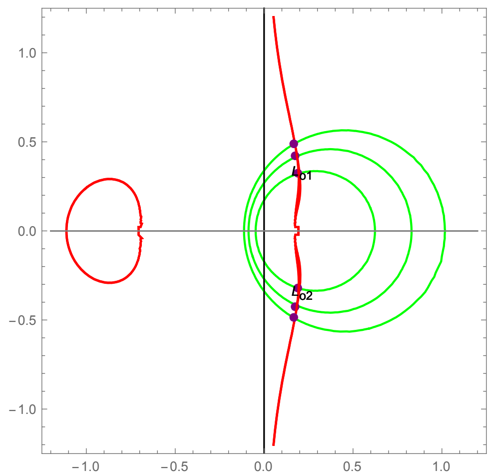

Fig. 8. The curves on $\bar{x}\bar{z}$ plane of $\mu = 0.1$, $A = 0.0001$ and $q = -0.01, -0.02, -0.03$ and $f = 3\pi/2$.

3. Stability of Equilibrium Points

To study the stability of the equilibrium point denoted by (a_0, b_0, c_0) of an infinitesimal body, we displace it to the position (x, y, z) with a small displacement (u, v, w) from the point, such that $x = a_0 + u$, $y = b_0 + v$, $z = c_0 + w$ substituting this value in (1), we obtain the following linearized

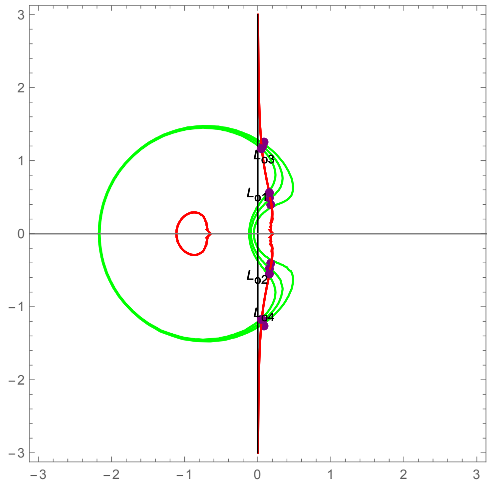

Fig. 9. The curves on $\bar{x}\bar{z}$ plane of $\mu = 0.1$, $A = 0.0001$ and $q = -0.01, -0.02, -0.03$ and $f = \pi$.

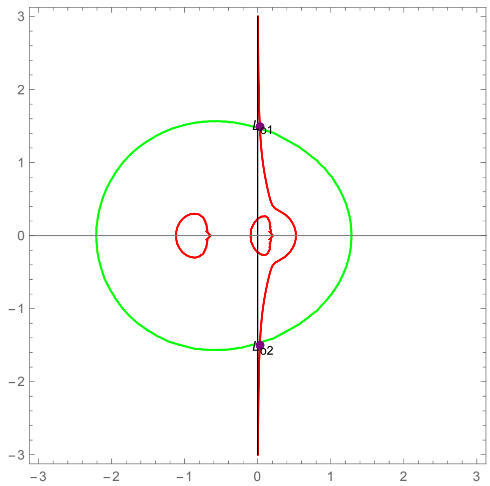

Fig. 10. The curves on $\bar{x}\bar{z}$ plane of $\mu = 0.1$, $A = 0.0001$ and $q = 0.01$ and $f = 0$.

system of equations:

$$\begin{aligned} u'' - 2v' &= \left[u\Omega^0_{xx} + v\Omega^0_{xy} + w\Omega^0_{xz}\right] \\ v'' + 2u' &= \left[u\Omega^0_{yx} + v\Omega^0_{yy} + w\Omega^0_{yz}\right] \\ w'' &= \left[u\Omega^0_{zx} + v\Omega^0_{zy} + w\Omega^0_{zz}\right] \end{aligned} \tag{12}$$

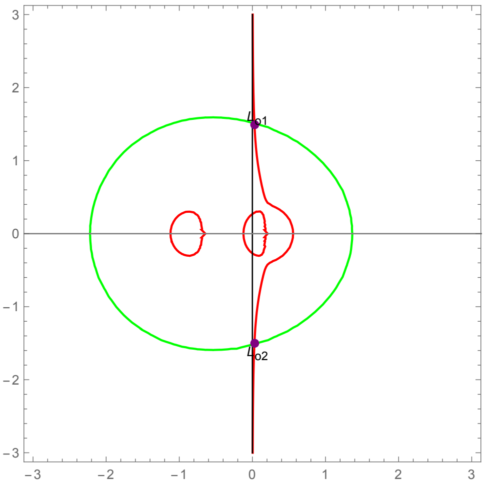

Fig. 11. The curves on $\bar{x}\bar{z}$ plane of $\mu = 0.1$, $A = 0.0001$ and $q = 0.01$ and $f = \pi/4$.

Here, Ω is defined in terms of ω as given by equation (5) and

$$\begin{aligned}
\omega_{\overline{xx}} =& 1 - \frac{1}{n^2}\Bigg[\frac{(1-2\mu)q}{r_1^3} - \frac{3(1-2\mu)(\overline{x}-\sqrt{3}\mu)^2 q}{r_1^5} + \frac{\mu}{r_2^3} + \frac{\mu}{r_3^3} \\
&- \frac{3\mu\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)^2}{r_2^5} - \frac{3\mu\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)^2}{r_3^5} + \frac{3\mu A_2}{2r_2^5} + \frac{3\mu A_1}{2r_3^5} \\
&- \frac{15\mu A_1\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)^2}{2r_3^7} - \frac{15\mu A_2\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)^2}{2r_2^7}\Bigg]
\end{aligned} \tag{13}$$

$$\begin{aligned}
\omega_{\overline{xy}} =& \frac{1}{n^2}\Bigg[\frac{3(1-2\mu)(\overline{x}-\sqrt{3}\mu)q\overline{y}}{r_1^5} + \frac{3\mu\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)^2(\overline{y}-1/2)}{r_2^5} \\
&+ \frac{3\mu\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)^2(\overline{y}+1/2)}{r_3^5} + \frac{15\mu(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2})^2(\overline{x}+1/2)A_1}{2r_3^7} \\
&+ \frac{15\mu\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)^2(\overline{y}-1/2)A_2}{2r_2^7}\Bigg]
\end{aligned} \tag{14}$$

$$\begin{aligned}
\omega_{\overline{yy}} =& 1 - \frac{1}{n^2}\Bigg[\frac{(1-2\mu)q}{r_1^3} - \frac{3(1-2\mu)\overline{y}^2 q}{r_1^5} + \frac{\mu}{r_2^3} - \frac{3\mu(\overline{y}-1/2)^2}{r_2^5} - \frac{3\mu(\overline{y}+1/2)^2}{r_3^5} \\
&+ \frac{\mu}{r_3^3} + \frac{3\mu A_1}{2r_3^5} + \frac{3\mu A_2}{2r_2^5} - \frac{3\mu A_1(\overline{y}+1/2)^2}{2r_3^7} - \frac{3\mu A_2(\overline{y}-1/2)^2}{2r_2^7}\Bigg] \\
\omega_{\overline{xz}} =& \frac{1}{n^2}\Bigg[\frac{3(1-2\mu)(\overline{x}-\sqrt{3}\mu)q\overline{z}}{r_1^5} + \frac{3\mu\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)^2(\overline{z})}{r_2^5} \\
&+ \frac{3\mu\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)(\overline{z})}{r_3^5} + \frac{15\mu(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2})(\overline{z})A_1}{2r_3^7} \\
&+ \frac{15\mu\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)^2(\overline{z})A_2}{2r_2^7}\Bigg] \\
\omega_{\overline{yz}} =& \frac{1}{n^2}\Bigg[\frac{(1-2\mu)\overline{z}q}{r_1^5} + \frac{3\mu(y-1/2)\overline{z}}{r_2^5} + \frac{3\mu(\overline{y}+1/2)\overline{z}}{r_3^5} \\
&+ \frac{15\mu A_1(\overline{y}+1/2)\overline{z}}{2r_3^7} - \frac{15\mu A_2(\overline{y}-1/2)\overline{z}}{2r_2^7}\Bigg] \\
\omega_{\overline{zz}} =& -\left[ecosf - \frac{1}{n^2}\left[\frac{(1-2\mu)q}{r_1^3} + \frac{3A\mu}{r_2^5} + \frac{2\mu}{r_2^3}\right]\right. \\
&- z^2\left[\frac{1}{n^2}\left[\frac{(1-2\mu)q}{r_1^5} + \frac{3A\mu}{2r_2^7} + \frac{2\mu}{r_2^5}\right]\right]
\end{aligned}$$

In order to investigate the stability of the out of plane equilibrium points, we introduce new variables as follows

$$P_x = \frac{du}{df}, P_y = \frac{dv}{df}, P_z = \frac{dw}{df}, \tag{15}$$

Assuming these values the differential equations take the forms:

$$\frac{du}{df} = P_{14}P_x + P_{15}P_y + P_{16}P_z + P_{11}u + P_{12}v + P_{13}w \tag{16}$$

$$\frac{dv}{df} = P_{24}P_x + P_{25}P_y + P_{26}P_z + P_{21}u + P_{22}v + P_{23}w \tag{17}$$

$$\frac{dw}{df} = P_{34}P_x + P_{35}P_y + P_{36}P_z + P_{31}u + P_{32}v + P_{33}w \tag{18}$$

$$\frac{dpx}{df} = P_{44}P_x + P_{45}P_y + P_{46}P_z + P_{41}u + P_{42}v + P_{43}w \tag{19}$$

$$\frac{dpy}{df} = P_{54}P_x + P_{55}P_y + P_{56}P_z + P_{51}u + P_{52}v + P_{53}w \tag{20}$$

$$\frac{dpz}{df} = P_{64}P_x + P_{65}P_y + P_{66}P_z + P_{61}u + P_{62}v + P_{63}w \tag{21}$$

where, the coefficients of the differential equations are $P_{14} = P_{25} = P_{36} = 1; P_{45} = -P_{54} = 2; P_{41} = \omega_{xx}; P_{42} = \omega_{xy}; P_{43} = \omega_{xz}; P_{51} = \omega_{yx}; P_{52} = \omega_{yy}; P_{53} = \omega_{yz}; P_{61} = \omega_{zx}; P_{62} = \omega_{zy}; P_{63} = \omega_{zz}$ and the values of all other coefficients are equal to zero. The coefficients P_{ij} i $\leq$ j, j ≤ 6 are periodic functions of f of period 2π considering the averaged system, where the averaged coefficients are given by

$$P_{ij}^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} P_{ij}(f)df \tag{22}$$

i $\leq$ j, j=1,2,3,4,5,6.

3.1. Stability of Planar (Collinear) Equilibrium Points

Using $\overline{y} = 0$,r_2=r_3 then A_1=A_2=A all second order derivative are given by

$$\begin{aligned}
\omega_{\overline{xx}} =& 1 - \frac{1}{n^2}\left[\frac{(1-2\mu)q}{r_1^3} - \frac{3(1-2\mu)(\overline{x}-\sqrt{3}\mu)^2 q}{r_1^5} + 2\frac{\mu}{r_2^3}\right. \\
& \left. - \frac{6\mu\left(\overline{x} + \frac{\sqrt{3}(1-2\mu)}{2}\right)^2}{r_2^5} - \frac{15\mu A\left(\overline{x} + \frac{\sqrt{3}(1-2\mu)}{2}\right)^2}{r_3^7} + \frac{3\mu A}{r_2^5}\right] \\
\omega_{\overline{xz}} =& \omega_{\overline{yx}} = \omega_{\overline{yz}} = 0 \\
\omega_{\overline{yy}} =& 1 - \frac{1}{n^2}\left[\frac{(1-2\mu)q}{r_1^3} + \frac{2A\mu}{r_2^3}\right] \\
\omega_{\overline{zz}} =& ecosf - \frac{1}{n^2}\left[\frac{(1-2\mu)q}{r_1^3} + \frac{2A\mu}{r_2^3}\right]
\end{aligned}$$

Now, to investigate the stability of the equilibrium points (a_0, b_0) in the first approximation,we derive the variational equation of motion in the coordinates as given by equation (15),(16),(18), (19) and (21). Where, after evaluating the values of the terms given by equation (15) collinear equilibrium points according to equation (12), we get:

$$P_{11}^{(0)} = P_{12}^{(0)} = P_{14}^{(0)} = P_{21}^{(0)} = P_{22}^{(0)} = P_{23}^{(0)} = P_{33}^{(0)} = P_{44}^{(0)} = 0;$$
$$P_{13}^{(0)} = 1, P_{24}^{(0)} = 1, P_{45}^{(0)} = 2, P_{54}^{(0)} = -2 \tag{23}$$

and the values of all other coefficients are equal to zero. Thus the characteristic equation for the system is :

$$\lambda^4 - Q\lambda^2 + R = 0 \tag{24}$$

where,

$$Q = P_{31}^{(0)} + P_{42}^{(0)} - 4$$
$$R = P_{31}^{(0)} \cdot P_{42}^{(0)} - P_{32}^{(0)} \cdot P_{41}^{(0)}. \tag{25}$$

The roots of characteristic equation (18) are given by:

$$\lambda = \pm \left[\frac{Q}{2} + \frac{(Q^2 - 4R)^{\frac{1}{2}}}{2} \right]^{\frac{1}{2}} \tag{26}$$

The characteristic roots will be purely imaginary if

$$Q < 0 \tag{27}$$

and

$$Q^2 - 4R \geq 0 \tag{28}$$

It is not feasible to determine the terms analytically, so we have tabulated the value of λ as shown in Table 1.

Table 1. Collinear Points and stability table for different value of Radiation Pressure and oblateness coefficient.

μ	A	q	L_1	$\lambda_{1,2}$	$\lambda_{3,4}$	L_2	$\lambda_{1,2}$	$\lambda_{3,4}$
0.1	0.0	1	-0.9286	$\pm 0.9925i$	$\pm 3.0682i$	1.057	± 1.3263	$\pm 2.4357i$
0.1	0.0	0.9	-0.8746	$\pm 1.1668i$	$\pm 3.2585i$	1.026	± 1.3471	$\pm 2.5134i$
0.1	0.0	0.8	-0.8403	$\pm 2.2702i$	± 3.3647	0.9888	± 1.3833	$\pm 2.627i$
0.1	0.0001	1	-0.9341	$\pm 0.9747i$	$\pm 3.0470i$	1.054	± 1.3364	$\pm 2.4514i$
0.1	0.0001	0.9	-0.8872	$\pm 1.1301i$	$\pm 3.3122i$	1.017	± 1.3790	$\pm 2.5660i$
0.1	0.0001	0.8	-0.8215	$\pm 1.3145i$	± 3.4255	0.988	± 1.3865	$\pm 2.6330i$

3.2. Stability of Non Planar Equilibrium Points

Using $\overline{y} = 0$, $r_2 = r_3$ then $A_1 = A_2 = A$ all second order derivative are given by

$$\begin{aligned}
\omega_{\overline{xz}} =& \frac{\overline{z}}{n^2}\left[\frac{(1-2\mu)(\overline{x}-\sqrt{3}\mu)^2 q}{r_1^5} + \frac{6\mu\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)}{r_2^5}\right. \\
&\left. - \frac{15\mu A\left(\overline{x}+\frac{\sqrt{3}(1-2\mu)}{2}\right)}{r_3^7}\right] \\
\omega_{\overline{xx}} =& \omega_{\overline{yx}} = 0 \\
\omega_{\overline{yz}} =& \frac{1}{n^2}\left[\frac{(1-2\mu)(\overline{z})q}{r_1^3}\right] \\
\omega_{\overline{yy}} =& 1 - \frac{1}{n^2}\left[\frac{(1-2\mu)q}{r_1^3} + \frac{2\mu}{r_2^3} + \frac{2A\mu}{r_2^3}\right] \\
\omega_{\overline{zz}} =& ecosf - \frac{1}{n^2}\left[\frac{(1-2\mu)q}{r_1^3} + \frac{2\mu}{r_2^3} + \frac{2A\mu}{r_2^3}\right]
\end{aligned}$$

The coefficients of characteristic equation for differential equation is given by

$$\lambda^6 + \kappa_0\lambda^4 + \kappa_1\lambda^2 + \kappa_2 = 0; \tag{29}$$

where,

$$\begin{aligned}
\kappa_0 =& S4 - P_{41}^{(0)} - P_{52}^{(0)} - P_{63}^{(0)}, \\
\kappa_1 =& P_{41}^{(0)}P_{52}^{(0)} - P_{41}^{(0)}P_{63}^{(0)} - P_{52}^{(0)}P_{63}^{(0)} - P_{42}^{(0)2} - P_{43}^{(0)} - P_{53}^{(0)2} - 4P_{63}^{(0)}, \\
\kappa_2 =& P_{43}^{(0)}P_{52}^{(0)2} + P_{63}^{(0)}P_{42}^{(0)2} + P_{41}^{(0)}P_{53}^{(0)2} - 2P_{42}^{(0)}P_{53}^{(0)}P_{43}^{(0)} - P_{52}^{(0)}P_{41}^{(0)}P_{63}^{(0)}.
\end{aligned} \tag{30}$$

Assuming $\lambda^2 = \rho$, the following cubic equation is obtained from equation

$$\rho^3 + \kappa_0\rho^2 + \kappa_1\rho + \kappa_2 = 0;$$
$$\kappa_0 > 0, \kappa_1 > 0, \kappa_2 > 0 and \Delta < 0,$$
$$\Delta = \frac{(2\kappa_0^3 - 9\kappa_0\kappa_1 + 27\kappa_2)^2 + 4(3\kappa_1 - {\kappa_0}^2)^3}{27}$$

now,

$$\kappa_0 > 0,$$

gives the condition

$$3 - \frac{1}{\sqrt{1-e^2}} > 0$$

that is $e < 0.866$. Further more, expanding the terms of the above inequalities, we get the following conditions:

$$0 < \kappa_1 <= \frac{(3 - \frac{1}{(\sqrt{1-e^2})})^2}{3} <= \frac{4}{3}$$

$$0 < \kappa_2 < \delta_1, \kappa_1 <= 1 \tag{31}$$
$$0 < \kappa_2 < \delta_2, \kappa_1 > 1 \tag{32}$$

where,

$$\delta_1 = \frac{(9\kappa_0\kappa_1 - 2\kappa_0^3 - 2((\kappa_0)^2 - 3\kappa_1)^3/2}{27} \tag{33}$$
$$\delta_2 = \frac{(9\kappa_0\kappa_1 - 2\kappa_0^3 - 2((\kappa_0)^2 - 3\kappa_1)^3/2}{27} \tag{34}$$

Thus the inequalities are utilized to define the stability region values of the various parameters such as oblateness, radiation pressure and so on. These conditions for stability are analogues to the conditions proposed by Ragos and Zagouras [12] in CRTBP when oblateness of the primaries and infinitesimal are neglected. The stability of the system will hold if the roots of equation (29) are purely imaginary.

4. Basin of Attraction

The basin of attraction of a point (attractor) is referred to the region from which each point after a number of iteration tends toward the point. These basins of attraction are mostly used to select the starting point for orbits around the equilibrium point. The possibility of getting stable orbit is high If the initial point is chosen from inside the region of attraction. However the initial point chosen among the boundary values shows chaotic behavior.We determine the basin of attraction of the planar (collinear) and non-planar (out of plane) libration points with the help of Newton Raphson method which was used extensively by Zotos [14, 15].

Table 2. The values of the characteristic roots for the pair equilibrium points $L_{O1,O2}$ for $f = 2\pi$, $A = 0.0001$, $e = .06$.

q	L_1 and L_2	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
-0.01	$(0.1968, 0.2942)$	$\pm 1.0995i$	± 2.760	$\pm 1.0995i$
-0.01	$(0.1968, -0.2942)$	$1.221 \pm 0.3457i$	$-1.221 \pm 0.3457i$	± 2.9271
-0.02	$(0.1893, 0.3841)$	$1.4664 \pm 0.4046i$	$-1.4664 \pm 0.4046i$	$\pm 1.708i$
-0.02	$(0.1893, -0.3841)$	$1.063 \pm 0.5902i$	$-1.063 \pm 0.5902i$	± 2.009
-0.03	$(0.1743, 0.4367)$	$1.3368 \pm 0.5875i$	$1.3368 \pm 0.5875i$	$\pm 1.6532i$
-0.03	$(0.1743, -0.4367)$	$0.9939 \pm 0.6704i$	$-0.9939 \pm 0.6704i$	± 1.7979

Table 3. The values of the characteristic roots for the pair equilibrium points $L_{O1,O2}$ for $f = 3\pi/2$, $A = 0.0001$, $e = 0.06$.

q	L_1 and L_2	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
-0.01	$(0.1978, 0.3242)$	$1.5260 \pm 0.06780i$	-1.5260 ± 0.06780	$\pm 1.65805i$
-0.01	$(0.1978, -0.3242)$	$1.0923 \pm 0.5065i$	$-1.0923 \pm 0.5065i$	± 2.1080
-0.02	$(0.174, 0.4217)$	$1.1603 \pm 0.6837i$	$-1.603 \pm 0.6837i$	$\pm 1.3844i$
-0.02	$(0.174, -0.4217)$	$0.9121 \pm 0.7220i$	$-0.9121 \pm 0.7220i$	± 1.4944
-0.03	$(0.1668, 0.4891)$	$1.0025 \pm 0.7743i$	$1.0025 \pm 0.7743i$	$\pm 1.3844i$
-0.03	$(0.1668, -0.4891)$	$0.7938 \pm 0.8055i$	-0.7938 ± 0.8055	± 1.306

4.1. Basin of Attraction for planar equilibrium points

We use iterative scheme for each equilibrium point in the $\overline{xy}$-plane as given by the following relation:

$$\overline{x}_{n+1} = \overline{x}_n - \frac{\Omega_{\overline{x}}\Omega_{\overline{yy}} - \Omega_{\overline{yy}}\Omega_{\overline{xy}}}{\Omega_{\overline{xx}}\Omega_{\overline{yy}} - \Omega^2_{\overline{yz}}} \tag{35}$$

$$\overline{y}_{n+1} = \overline{y}_n - \frac{\Omega_{\overline{x}}\Omega_{\overline{yx}} - \Omega_{\overline{y}}\Omega_{\overline{yy}}}{\Omega_{\overline{xx}}\Omega_{\overline{yy}} - \Omega^2_{\overline{xy}}} \tag{36}$$

All the initial condition are provided such as equilibrium point ($\overline{x}$,$\overline{y}$), mass parameter and oblateness coefficient. We present the basin of attraction using Newton Raphson method for the equilibrium points in the restricted four body problem. The colour coded diagrams in the ($\overline{x}$-$\overline{y}$) plane are plotted for different value of f.

4.2 Basin of Attraction for non-planar equilibrium points

We determine the basin of attraction of the planar (collinear) and non-planar (out of plane) libration points with the help of Newton Raphson method

Table 4. The values of the characteristic roots for the pair equilibrium points $L_{O1,O2}$ for $f=\pi$, $A=0.0001$, $e=0.06$.

q	$L_{O1} and L_{O2}$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
-0.01	$(0.1779, 0.3969)$	$0.8904 \pm 0.7816i$	-0.8904 ± 0.7816	$\pm 1.2248i$
-0.01	$(0.1779, -0.3969)$	$0.7493 \pm 0.8137i$	$-0.7493 \pm 0.8137i$	± 1.1239
-0.01	$(0.08668, 1.255)$	$0.0 \pm 1.189i$	$0.0 \pm 0.2241i$	$\pm .8051i$
-0.01	$(0.08668, -1.255)$	$\pm 0.8071i$	$0.0 \pm 1.189i$	$\pm .3585$
-0.02	$(0.1631, 0.4826)$	$0.7991 \pm 0.819i$	$-0.7991 \pm 0.8191i$	$\pm 1.2132i$
-0.02	$(0.1631, -0.4826)$	$0.6492 \pm 0.8652i$	$-0.6492 \pm 0.8652i$	± 1.082
-0.02	$(0.06337, 1.197)$	$0.0 \pm 0.2780i$	$-0.0 \pm 0.8034i$	$\pm 1.189i$
-0.02	$(0.06337, 1.197)$	$0 \pm 0.7984i$	$-0.0 \pm 1.1968i$	$\pm ..3913i$
-0.03	$(0.1515, 0.5595)$	$0.6800 \pm 0.8414i$	$0.6800 \pm 0.8414i$	$\pm 1.595i$
-0.03	$(0.1515, -0.5595)$	$0.6800 \pm 0.8414i$	-0.6800 ± 0.8414	± 1.1595
-0.03	$(0.0418, 1.165)$	$0.0 \pm 0.3195i$	$0.0 \pm 0.8095i$	$\pm 1.179i$
-0.03	$(0.0418, -1.165)$	$0.4219 \pm 0.8156i$	-0.4219 ± 0.81564	± 0.4219

which was used extensively by Zotos [14, 15]. We use iterative scheme for each equilibrium point in the $\overline{xz}$-plane as given by the following relation:

$$\overline{x}_{n+1} = \overline{x}_n - \frac{\Omega_{\overline{x}}\Omega_{\overline{zz}} - \Omega_{\overline{zz}}\Omega_{\overline{xz}}}{\Omega_{\overline{xx}}\Omega_{\overline{zz}} - \Omega^2_{\overline{xz}}} \tag{37}$$

$$\overline{z}_{n+1} = \overline{z}_n - \frac{\Omega_{\overline{x}}\Omega_{\overline{zx}} - \Omega_{\overline{z}}\Omega_{\overline{zz}}}{\Omega_{\overline{xx}}\Omega_{\overline{zz}} - \Omega^2_{\overline{xz}}} \tag{38}$$

All the initial conditions are provided such as equilibrium point ($\overline{x}$,$\overline{z}$), mass parameter and oblateness coefficient. We present the basin of attraction using Newton Raphson method for the equilibrium points in the restricted four body problem. The colour coded diagrams in the ($\overline{x}$-$\overline{z}$) plane are plotted for different value of q and f.

5. Conclusion and Discussion

The position and stability of collinear and non-planar equilibrium points in the elliptical restricted four body problem, where one of the primary is radiating and other two are oblate spheroid, moving in elliptic orbit around their common center of mass, has been investigated. We found that the location of planar (collinear) and non-planar (out of plane) Libration points are affected by parameters perturbation forces such as oblateness and radiation pressure.

We observed that for fixed value of oblateness parameter A, when radiation pressure is increased from 0.8 to 1.0 both planar equilibrium points L_1 and

Fig. 12. Fractal basin for planar equilibrium point $\mu = 0.1$, $A = 0.0001$ and $q = 0.01$ and $f = \pi/2$.

Fig. 13. Fractal basin for planar equilibrium point $\mu = 0.1$, $A = 0.0001$ and $q = 0.01$ and $f = \pi$.

L_2 shift away from the origin. Whereas when the oblateness is decreased the shift is toward the origin.

The existence and position of non-planar (out of plane) equilibrium points depend on true anomaly f and radiation pressure q. For $A = 0.0001$, $q = -0.01$ to $q = -0.03$ and $f = (2n+1)\pi$ there exist four out-of-plane equilibrium points and for $A = 0.0001$, $q = -0.01$ to $q = -0.03$ and for $f = (2n+1)\pi/2$ there exists two out-of-plane equilibrium points. For $A = 0.0001$, $q = -0.01$ to $q = -0.03$ and $f = 2n\pi$ there exists two out-of-plane equilibrium points

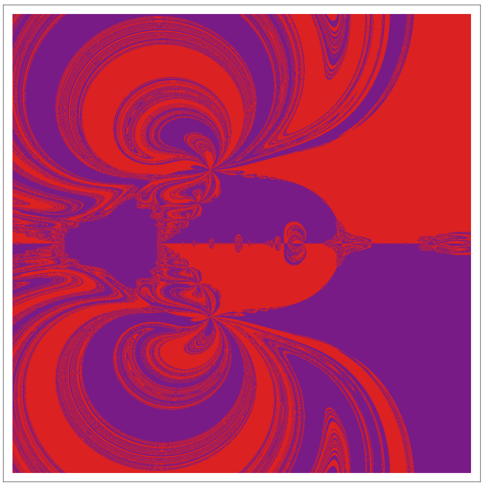

Fig. 14. Fractal basin for non-planar equilibrium point $\mu = 0.1$, $A = 0.0001$ and $q = -0.01$ and $f = 0$.

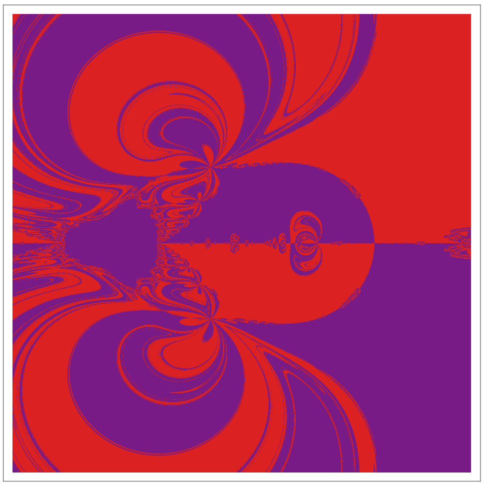

Fig. 15. Fractal basin for non-planar equilibrium point $\mu = 0.1$, $A = 0.0001$ and $q = -0.02$ and $f = 0$.

whereas for $A = 0.0001$, $q = 0.01$ to $q = 0.08$ and $f = \pi/4$ and there exists two out-of-plane equilibrium points.

These points are shifted toward the origin and $f = \pi/2$ and or $A = 0.0001$, $q = 0.01$ to $q = 0.08$ (positive value) is no equilibrium points whereas for $q = .01$ to 0.6 and $f = 0$ their exist two out of plane equilibrium points. We also observed that there is very little difference in the position of out of plane equilibrium points for different values of the oblateness coefficient.

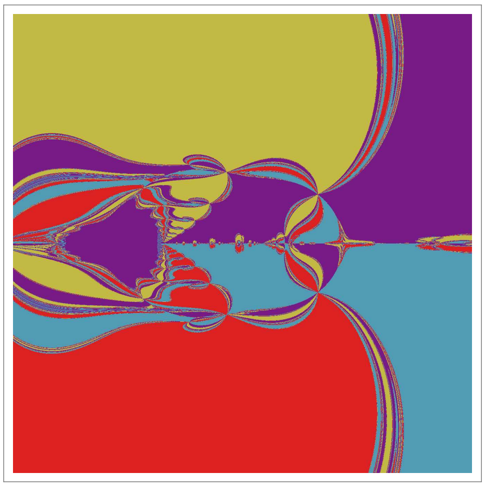

Fig. 16. Fractal basin for non-planar equilibrium point $\mu = 0.1$, $A = 0.0001$ and $q = -0.01$ and $f = \pi$.

Fig. 17. Fractal basin for non-planar equilibrium point $\mu = 0.1$, $A = 0.0001$ and $q = -0.02$ and $f = \pi$.

The stability criteria for both planar (collinear) and non-planar (out of plane) equilibrium points were established and were numerically studied by tabulating their eigen values. It was also shown that the collinear and non-planar equilibrium points are unstable.

The basin of attraction for collinear equilibrium point in the xy-plane was plotted for $f = \pi/2$, $q = 0.01$ and $A = 0.0001$ also for $f = \pi$, $q = 0.01$ and $A = 0.0001$. It was observed that thre are no major changes in fractal basin for changes in true anomaly.

The basin of attraction for out of plane equilibrium point in the xz-plane was plotted for $f = 0$, $q = -0.01$ and $q = -0.02$ also for $f = \pi$ $q = -0.01$ and $q = -0.02$. It was observed that as radiation factor q decreases the basin of attraction points enlarges and its vagueness decreases.

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