Heterogeneous primary in the restricted three-body problem with modified Newtonian potential of secondary body

Abdullah A. Ansari
International Center for Advanced Interdisciplinary Research (ICAIR),
Sangam Vihar, New Delhi, India.
icairndin@gmail.com

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Abstract. The aim of the paper is to study the motion of a test particle and its property of stability. The particle is moving under the influence of a heterogeneous primary having N-layers with different densities. The particle is also under the influence of the secondary body which is producing the modified Newtonian potential. The system is perturbed by the small perturbations in Coriolis as well as centrifugal forces. We evaluate the equations of motion of the test particle under the influence of the above said perturbations. From the above system of equations of motion, we reveal locations of stationary points as well as their stability, analytically and numerically.

Key words: Heterogeneous primary body, Secondary body, Modified Newtonian potential, Stability

1 Introduction

The circular restricted three-body problem is widely studied due to its wide applications. It is used when the two massive bodies (known as primary and secondary bodies) are moving in circular orbits around their common center of mass in the same plane. These two bodies impose the newtonian forces on the third smallest body (known as infinitesimal body). The third body is not having any affect on the other two massive bodies. (Szebehely [1967], Henon [1969], Zagouras [1991], Murray et al. [1999], Pal [2014], Barrabes et al. [2015], Pathak et al. [2017], Mia [2020].)

Celestial mechanics is the branch of applied mathematics and mathematical physics where researchers are investigating the motion of the minor body under the influence of the other bigger celestial bodies. These motions form the various types of configurations such as restricted problems (three-body, four-body, five-body as well as N-body) either in circular or elliptical orbits with many special configurations (Copenhagen, Robes, Hill's etc.). These problems with varied number of configurations are attracting many researchers where they are considering one of these configurations with the supposition of different perturbations. Some of the related works are: Palmore [1973, 1975, 1976], Sharma et al. [1976], Simo [1978], Hallan et al. [2001], Beevi et al. [2012], Ansari [2018], Abouelmagd et al. [2019], Ansari et al. [2019], Abouelmagd et al. [2020].

The restricted problems take in account the shapes of the bigger celestial bodies as irregular (triaxial rigid bodies, spherical bodies, oblate bodies, prolate bodies, heterogeneous bodies having N - layers with different densities, finite straight segments, cylindrical bodies) due to which the motion of the smallest body is perturbed (Abouelmagd et al. [2013], Singh et al. [2015], Ansari [2017], Ansari et al. [2019], Ansari [2020], Ansari et al. [2020]).

Some researchers and scientists have conducted their studies by supposing various types of force factors (such as Newtonian forces, modified Newtonian forces, Manev-type potential, effect of charge, solar radiation pressure, Yarkovsky effects, Yukawa effects, albedo effects, variable mass, resonance, viscous force, asteroids belt, Coriolis and centrifugal forces, etc.) due to which the motion properties of the smallest body are affected (Bhatnagar et al. [1978], Singh et al. [1985], Kokubun [2004], Abdulraheem et al. [2008], Lukyanov [2009], Ershkov [2012], Abouelmagd et al. [2012], Ansari et al. [2019], Zotos et al. [2020], Ansari et al. [2020], Ferdaus [2020]).

From the literature review, it is clear that all the celestial bodies are having irregular shapes with many layers and they are affected by many perturbing forces. Till now no researchers have studied the restricted three-body problem with the combination of heterogeneous primary and modified newtonian potential by secondary and this subject needs to be explored. Therefore, in this paper we have considered the restricted three-body configuration where the primary body has a heterogeneous shape and N-layers with different densities for each layer, while the secondary is a point mass which is producing the modified Newtonian potential. We have also considered that the system is perturbed by the small perturbations in the Coriolis and centrifugal forces. Further, we have investigated the effect of these perturbations on the motion properties of the smallest body (the third infinitesimal body).

The paper is organized in 7 parts. The literature review is presented in the introduction (section 1). The potentials between the bodies are introduced in the Gravitational potential used in the problem (section 2). In section 3, Equations of motion, we have evaluated the equations of motion with the various perturbations. In section 4, Stationary points, we have determined the locations of stationary points analytically in the two subsections 4.1 and 4.2, The collinear stationary points and Triangular stationary points, respectively. In section 5, Stability, we have performed the stability of the stationary points with the various perturbations analytically. Further in section 6, Numerical studies, there are two subsections 6.1 and 6.2 illustrating numerically the locations of stationary points and their stability respectively. Finally the conclusion is drawn in section 7.

2 Gravitational Potential used in this problem

Let the primary body m_1 be heterogeneous with N-layers having different densities ρ_i and axes $(a_i\,,\, b_i\,,\, c_i)$, $i = 1, 2, 3, - - - -, N$, while the third body with mass m is a point mass. And also

$$\rho_i \,<\, \rho_{i+1},\, a_i \,<\, a_{i+1},\, b_i \,<\, b_{i+1},\, c_i \,<\, c_{i+1},\, i = 1, 2, 3, - - - - - - -, N - 1.$$

Then the gravitational potential between m_1 and m with gravitational constant G, will be (see Ansari et al. [2020])

$$U_1 = -\frac{G\, m_1\, m}{r_1} - \frac{G\, m}{2\, r_1^3}\left[h_{11} - \frac{3}{r_1^2}\left(h_{12}\, y^2 + h_{13}\, z^2\right)\right], \qquad (1)$$

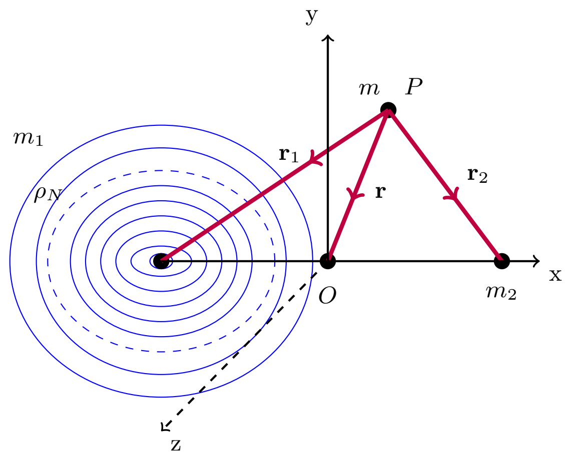


Fig. 1. Geometric plan for the problem with heterogeneous primary and modified Newtonian potential of secondary

where,

$$h_{11} = \frac{4\,\pi}{3} \sum_{i=1}^{N} \frac{1}{5} (\rho_i - \rho_{i+1}) a_i \, b_i \, c_i (2\, a_i^2 - b_i^2 - c_i^2),$$

$$h_{12} = \frac{4\,\pi}{3} \sum_{i=1}^{N} \frac{1}{5} (\rho_i - \rho_{i+1}) a_i b_i c_i (a_i^2 - b_i^2),$$

$$h_{13} = \frac{4\,\pi}{3} \sum_{i=1}^{N} \frac{1}{5} (\rho_i - \rho_{i+1}) a_i b_i c_i (a_i^2 - c_i^2).$$

Further, we assumed that the secondary body of mass m_2 is producing modified Newtonian potential with perturbing parameter ϵ. Therefore the gravitational potential between the secondary body and the third body of mass m with separation distance r_2 is (see Abouelmagd [2018, 2020]):

$$U_2 = -\frac{G\, m_2\, m\, r_2}{r_2^2 + \epsilon}, \tag{2}$$

3 Equations of motion

This problem contains three masses, m_1 (primary body), m_2 (secondary body) and m (the smallest body), where m_1 is a heterogeneous body with $N-$layers having different densities ρ_N, and m_2 is a point mass which is producing modified Newtonian potential. Both primary and secondary are moving in circular orbits around their common center of mass which is taken as origin O with radii ℓ_1 and ℓ_2, respectively, and $\ell_2 > \ell_1$. The system is also perturbed by the small perturbations in the Coriolis and centrifugal forces with the parameters α and β, respectively. The third smallest body of mass m is moving in space and follows the synodic coordinate system which is rotating with angular velocity n (Fig. 1).

- The total gravitational potential exerted by both bodies (primary and secondary) on the third body will be:

$$V = -\frac{G\,m_1\,m}{r_1} - \frac{G\,m}{2\,r_1^3}\left[h_{1\,1} - \frac{3}{r_1^2}\,h_{1\,2}\,y^2\right] - \frac{G\,m_2\,m\,r_2}{(r_2^2+\epsilon)}.$$

For the dimensionless variables, we have $m_1 + m_2 = 1$, $G = 1$ and the separation distance R between the primaries is unity i.e $R = \ell_1 + \ell_2 = 1$, and also $\mu = \dfrac{m_2}{(m_1 + m_2)}$. Hence $m_1 = 1 - \mu$ and J_j are the dimensionless quantities of h_{1j} for j = 1, 2. Therefore the equations of motion of the third small body in the cartesian coordinate will be as follows:

$$\begin{aligned} \ddot{x} - 2\,n\,\alpha\,\dot{y} &= U_x, \\ \ddot{y} + 2\,n\,\alpha\,\dot{x} &= U_y, \end{aligned} \tag{3}$$

with

$$n^2 = 1 - 3\,\epsilon + \frac{3}{2}\left(\frac{J_1}{1-\mu}\right),$$

$$U = \frac{n^2\,\beta}{2}(x^2+y^2) + \frac{(1-\mu)}{r_1} + \frac{1}{2\,r_1^3}\left[J_1 - \frac{3}{r_1^2}\,J_2\,y^2\right] + \frac{\mu\,r_2}{r_2^2+\epsilon}, \tag{4}$$

$$\begin{aligned} U_x = n^2\,\beta\,x &- \frac{\mu\,(x-1+\mu)\,(r_2^2-\epsilon)}{r_2\,(r_2^2+\epsilon)^2} - \frac{(1-\mu)(x+\mu)}{r_1^3} \\ &- \frac{3\,(x+\mu)}{2\,r_1^5}\left(J_1 - \frac{5}{r_1^2}\,J_2\,y^2\right), \end{aligned} \tag{5}$$

$$U_y = \left(n^2\,\beta - \frac{\mu\,(r_2^2-\epsilon)}{r_2\,(r_2^2+\epsilon)^2} - \frac{(1-\mu)}{r_1^3} - \frac{3}{2\,r_1^5}\left(J_1 + 2\,J_2 - \frac{5}{r_1^2}\,J_2\,y^2\right)\right)y, \tag{6}$$

$$r_1^2 = (x+\mu)^2 + y^2 \;\; \& \; r_2^2 = (x-1+\mu)^2 + y^2. \tag{7}$$

4 Stationary points

For the stationary points, we have to put zero to all the derivative with respect to time in the system (3), hence

$$\begin{aligned} n^2\,\beta\,x \,-\, \frac{\mu\,(x+\mu-1)\,(r_2^2-\epsilon)}{r_2\,(r_2^2+\epsilon)^2} \,-\, \frac{(1-\mu)(x+\mu)}{r_1^3} \\ -\,\frac{3\,(x+\mu)}{2\,r_1^5}\left[J_1-\frac{5}{r_1^2}\,J_2\,y^2\right]=0, \end{aligned} \tag{8}$$

$$\left(n^2\,\beta \,-\, \frac{\mu(r_2^2-\epsilon)}{r_2\,(r_2^2+\epsilon)^2} \,-\, \frac{(1-\mu)}{r_1^3} \,-\, \frac{3}{2\,r_1^5}\left(J_1+2\,J_2\,-\,\frac{5}{r_1^2}\,J_2\,y^2\right)\right)y=0, \tag{9}$$

After solving equations (8) and (9), we can find the locations of stationary points in two cases.

4.1 Case-I: Locations of collinear stationary points:

Collinear stationary points can be obtained from equation (8) by taking $x \neq 0$, $y = 0$.

$$\begin{aligned} n^2\,\beta\,x \,-\, \frac{\mu\,(x+\mu-1)\,((x+\mu-1)^2-\epsilon)}{|x+\mu-1|\,|((x+\mu-1)^2+\epsilon)|^2} - \frac{(1-\mu)(x+\mu)}{|x\,+\,\mu|^3} \\ -\,\frac{3\,(x+\mu)\,J_1}{2\,|x\,+\,\mu|^5}=0, \end{aligned} \tag{10}$$

After putting $x+\mu=\xi$ and more simplification of Eq. (10), we get the new equation in ξ of ninth degree as:

$$\begin{aligned} (91\,\epsilon\,-\,6\,)\,\xi^9+(55\,\epsilon\,-\,5\,)\,\xi^8+\,(30\,\epsilon\,-\,4\,)\,\xi^7 \\ +(14\,\epsilon\,-\,3\,)\,\xi^6+\left(-2\,+5\,\epsilon\,+\frac{\beta}{\mu}\left(1-3\,\epsilon+\frac{3\,J_1}{2\,(1-\mu)}\right)\right)\xi^5 \\ +\left(-\,1-\beta\,+\,\epsilon\,+3\,\beta\,\epsilon-\frac{3\,J_1\,\beta}{2\,(1-\mu)}\right)\xi^4+(1\,-\,\frac{1}{\mu})\,\xi^2-\frac{3\,J_1}{2\,\mu}=0 \end{aligned} \tag{11}$$

Eq. (11) is a ninth degree equation, therefore it will give nine values of ξ. To determine the locations of collinear stationary points, we divide x-axis in three different subintervals as $x\in(-\infty,-\mu)$, $x\in(-\mu,(1-\mu))$ and $x\in(1-\mu,\infty)$. From further investigations, we got one real value of x within each interval. Hence out of nine roots of Eq. (11), six will be complex and three will be real values. These three values will be the location of collinear stationary points, in general these are denoted as L_1, L_2 and L_3.

4.2 Case-II: Locations of triangular stationary points:

Non-collinear stationary points can be obtained from equations (8) and (9) when $x \neq 0$ and $y \neq 0$. If both the massive bodies are point masses, then $r_1 = 1, r_2 = 1$ will be the solution. But in our case the primary is having heterogeneous shape, hence let us assume the solution to be

$$r_1 = 1 + \delta_1, r_2 = 1 + \delta_2,\ \delta_1 << 1, \delta_2 << 1,$$

Then, from Eqs. (7), we get

$$\begin{cases} x = \frac{1}{2} - \mu + \delta_1 - \delta_2, \\ y = \pm \frac{\sqrt{3}}{2}\left(1 + \frac{2}{3}(\delta_1 + \delta_2)\right). \end{cases} \tag{12}$$

Putting the values of r_1, r_2, x and y in Eqs. (8) and (9), as well as neglecting the higher powers of δ_i (i = 1, 2), we obtain

$$\delta_1 = \left(\frac{\begin{array}{c} -\mu + \beta\,\mu + \mu^2 - \beta\,\mu^2 + \left(\frac{1 - 2\,\beta + \beta^2}{3}\right) \\ +\epsilon\left(2\,\beta - 2\,\beta^2 - \frac{4\,\mu + 5\,\beta\,\mu + 2\,\mu^2 - 11\,\beta\,\mu^2}{3}\right) \\ +J_1\left(1 - \beta - \frac{\beta}{1-\mu} + \frac{\beta^2}{1-\mu} - \frac{3\,\mu}{2} + \frac{3\,\beta\,\mu}{2\,(1-\mu)}\right. \\ \left. - \frac{3\,\beta\,\mu^2}{2(1-\mu)}\right) + J_2\left(-\frac{11}{4} + \frac{11\,\beta}{4} + \frac{53\,\mu}{8} - \frac{5\,\beta\,\mu}{2}\right) \end{array}}{\begin{array}{c} -1 + \beta + 3\,\mu - 3\,\mu^2 + \epsilon\left(-3\,\beta + 2\,\mu^2 + \frac{5\,\mu - 2\,\beta\,\mu}{3}\right) \\ +J_1\left(-4 + \frac{5\,\beta}{2} + \frac{3\,\beta}{2(1-\mu)} + \frac{15\,\mu}{2}\right) \\ +J_2\left(\frac{35}{4} - \frac{25\,\beta}{8} - \frac{195\,\mu}{8}\right) \end{array}} \right) \tag{13}$$

&

$$\delta_2 = \left(\frac{\begin{array}{c} 1 - 2\,\beta + \beta^2 - 3\,\mu + 3\,\beta\,\mu + 3\,\mu^2 - 3\,\beta\,\mu^2 \\ -\,\epsilon\left(-\,6\,\beta + 6\,\beta^2 - 3\,\mu + 3\,\beta\,\mu + 9\,\mu^2 - 9\,\beta\,\mu^2\right) \\ -\,J_1\left(-3 + 3\,\beta + \dfrac{3\,\beta}{(1-\mu)} - \dfrac{3\,\beta^2}{(1-\mu)} + \dfrac{15\,\mu}{2} \right. \\ \left. -\dfrac{15\,\beta\,\mu}{2} - \dfrac{9\,\beta\,\mu}{2\,(1-\mu)} + \dfrac{9\,\beta\,\mu^2}{2\,(1-\mu)}\right) \\ -\,J_2\left(\dfrac{21}{4} - \dfrac{3\,\beta}{4} - \dfrac{171\,\mu}{8} + \dfrac{135\,\beta\,\mu}{8}\right) \end{array}}{\begin{array}{c} 1 - \beta - 3\,\mu + 3\,\mu^2 \\ +\,\epsilon\left(3\,\beta - \dfrac{5\,\mu}{3} + \dfrac{2\,\beta\,\mu}{3} - 2\,\mu^2\right) \\ +\,J_1\left(4 - \dfrac{5\,\beta}{2} - \dfrac{3\,\beta}{2(1-\mu)} - \dfrac{(15\,\mu)}{2}\right) \\ +\,J_2\left(-\dfrac{35}{4} + \dfrac{25\,\beta}{8} + \dfrac{195\,\mu}{8}\right) \end{array}} \right). \tag{14}$$

Eq. (12) is representing the locations of triangular stationary points, where the positive sign of y represents L_4 and the negative sign of y represents L_5.

5 Stability of stationary points

We can write the variational equation of system (3) by putting $x = L_x + \xi$, $\xi << 1$ and $y = L_y + \eta$, $\eta << 1$. We obtained

$$\begin{aligned} \ddot{\xi} \,-\, 2\,n\,\alpha\,\dot{\eta} &= U^0_{xx}\,\xi \,+\, U^0_{xy}\,\eta, \\ \ddot{\eta} \,+\, 2\,n\,\alpha\,\dot{\xi} &= U^0_{yx}\,\xi \,+\, U^0_{yy}\,\eta. \end{aligned} \tag{15}$$

The superscript 0 indicates that the derivatives are to be calculated at the corresponding stationary point.

Let us suppose the trial solution as

$$\xi = C_1\,e^{\lambda\,t}, \eta = C_2\,e^{\lambda\,t}.$$

Using the values of the trial solution in Eq. (15), we have

$$C_1\,\left(\lambda^2\,-\,U^0_{xx}\right)+\,C_2\left(-\,2\,n\,\alpha\,\lambda-U^0_{xy}\right)=0,$$
$$C_1\,\left(2\,n\,\alpha\,\lambda-U^0_{yx}\right)+\,C_2\,\left(\lambda^2\,-\,U^0_{yy}\right)=0. \tag{16}$$

We will have a non-trivial solution if

$$\begin{vmatrix} \lambda^2\,-\,U^0_{xx} & -\,2\,n\,\alpha\,\lambda-U^0_{xy} \\ 2\,n\,\alpha\,\lambda-U^0_{yx} & \lambda^2\,-\,U^0_{yy} \end{vmatrix}=0.$$

From the above determinant, we have

$$f(\lambda)=\lambda^4+A_2\,\lambda^2+A_1\,\lambda+A_0, \tag{17}$$

with

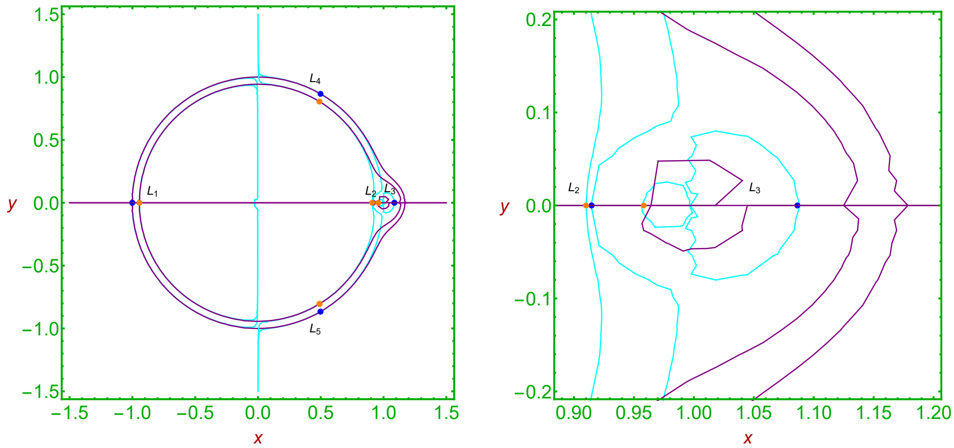
$$A_2=4\,n^2\,\alpha^2\,-\,U^0_{xx}\,-\,U^0_{yy},$$
$$A_1=2\,n\,\alpha\left(U^0_{xy}\,-\,U^0_{y\,x}\right), \tag{18}$$
$$A_0=U^0_{xx}\,U^0_{yy}\,-\,U^0_{xy}\,U^0_{yx}.$$

The classical circular restricted three-body problem has five stationary points out of which three are collinear and two are triangular stationary points. The three-collinear stationary points are always unstable while triangular stationary points are stable for some values of mass ratio μ (for more detail see Szebehely [1967]). Now, if $\lambda\to\infty$, then $f(\lambda)\to\infty$, and $f(0)=A_0$. Here the stability of the stationary points will depend on the value of A_0, i.e. if $A_0<0$, then there will be at least one positive root, so the stationary points will be unstable.

6 Numerical Studies

6.1 Location of stationary points

To confirm and compare our analytical work for the locations of stationary points, we have illustrated the numerical work in two cases: the classical case (μ = 0.0019; $\alpha=1$; $\beta=1$; $J_1=0$; $J_2=0$; $\epsilon=0$) and the perturbed case (μ = 0.0019; $\alpha=1.2$; $\beta=1.2$; $J_1=0.001$; $J_2=0.00012$; $\epsilon=0.002$). In both the cases we got five stationary points out of which three are collinear (L_1, L_2 and L_3), while two are triangular stationary points (L_4 and L_5). From figure 2(a), we observed that in the perturbed case all the stationary points are moving towards the origin. This movement near L_2 and L_3 can be clearly seen in figure 2(b) which is zoomed part of figure 2(a).



(a) Locations for the classical case (blue) and the perturbed case (orange).

(b) Zoomed part of figure 2(a) near L_2 and L_3.

Fig. 2. Locations of stationary points in $x - y$–plane.

6.2 Stability Analysis

Using Eq. (17), we illustrated the nature of stationary points. We solved Eq. (17) numerically for all mentioned parameters to evaluate the characteristic roots corresponding to each stationary point, which we give and represented in Table 1. From this table we observed that corresponding to the collinear stationary points, for both cases (classical case and perturbed case), we get at least one positive real root or positive real part of the complex roots. Therefore, all the collinear stationary points are unstable. On the other-hand, corresponding to triangular stationary points, for both cases, we obtained all the characteristic roots as purely imaginary. Hence both the triangular stationary points in both cases are stable.

7 Conclusion

This study concerns the stability properties of the motion of a test particle which is moving under the influence of the heterogeneous primary having N-layers with different densities ρ_i of each layers, and a point mass of the secondary body producing the modified Newtonian potential with modified parameter ϵ. The obtained equations of motion are different from the classical case by the perturbation parameters α, β, J_1, J_2 and ϵ. The effect of these parameters can be easily seen in the analytical and numerical studies of the locations and stability of the stationary points. We got five stationary points, out of which three are collinear and two are triangular stationary points as in the classical case. We also observed that all the stationary points are moving towards the origin under the perturbation effects. We further observed in the study of stability that the collinear stationary points are always unstable, while triangular stationary points are stable as in the classical case.

Table 1. The characteristic roots corresponding to stationary points in $x-y$-plane and their nature in two cases

Cases	*Stationary Point* $x-Co.$	$y-Co.$	*Roots*	*Nature*
Classical Case	$-\ 1.00047$	0.00000	$\pm 1.00260\, i$ $\pm$ **0.08845**	*Unstable*
	0.91471	0.00000	$\pm 2.20686\, i$ $\pm$ **2.72821**	*Unstable*
	1.08639	0.00000	$\pm 1.95430\, i$ $\pm$ **2.31375**	*Unstable*
	0.49810	± 0.86602	$\pm 0.11388\, i$ $\pm 0.99349\, i$	*Stable*
Perturbed Case	$-\ 0.94379$	0.00000	$\pm 1.46564\, i$ $\pm$ **0.06197**	*Unstable*
	0.91010	0.00000	$\pm 2.05276\, i$ $\pm$ **1.23108**	*Unstable*
	0.95820	0.00000	$\pm 4.02490\, i$ $\pm$ **1.03172**	*Unstable*
	0.48991	± 0.80450	$\pm 0.09490\, i$ $\pm 1.46205\, i$	*Stable*

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