Exact Kantowski-Sachs Solutions in Metric-Torsion Gravity

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Abstract. We investigate Kantowski–Sachs cosmological models within the metric-torsion gravity framework developed by Yaremenko (2014). Assuming the minimal torsion compatible with local rotational symmetry, we derive the full set of field equations from the generalized Einstein–Hilbert action. The torsion is described by three independent functions of time: vector components $f(t)$ and $g(t)$, and an axial component $h(t)$. We obtain modified Friedmann equations that couple the scale factors $X(t)$ and $Y(t)$ to these torsion functions. Power-law solutions are found and analysed. The expansion scalar, shear scalar and volume are computed; it is shown that torsion can modify the dynamics and, for certain parameter ranges, may remove the initial singularity. Our results provide a concrete realisation of how torsion influences anisotropic cosmology and offer testable predictions for early-universe scenarios.

Key words: Kantowski–Sachs Metric-torsion gravity Anisotropic cosmology Torsion Early universe

1. Introduction

The standard cosmological model, based on general relativity (GR) and the Friedmann–Lemaître–Robertson–Walker (FLRW) metric, has achieved remarkable success in describing the large-scale evolution of the universe. However, the early universe may have passed through an anisotropic phase before reaching the highly isotropic state observed today [1969, 1975]. Anisotropic cosmological models provide a valuable testing ground for gravitational theories beyond GR and offer insights into the isotropisation process that characterises our present universe.

Among anisotropic cosmologies, the Kantowski–Sachs (KS) model [1966] occupies a special place. It describes a spatially homogeneous but anisotropic universe with a preferred radial direction and spherical symmetry in the angular sector. This geometry exhibits local rotational symmetry (LRS) and has been extensively studied in general relativity [1977, 1985] as well as in various modified gravity theories [1999, 2007, 2010]. The KS model is particularly interesting because it can represent either a closed universe with a topological structure different from FLRW or an anisotropic phase preceding isotropisation.

One of the most natural extensions of Einstein's theory is metric-torsion gravity, where the spacetime connection is not symmetric and the torsion tensor becomes an independent dynamical field [1976, 1995, 2002]. In this framework, geometry is jointly generated by the metric g_{ik} (determining distances) and the torsion tensor $S^{i}{}_{jk}$ (describing the non-closure of parallelograms). This formalism, often referred to as Einstein–Cartan theory in its simplest form [1972, 1973], generalises GR by allowing the connection to have an antisymmetric part while maintaining metric compatibility.

Torsion arises naturally from the first-order (Palatini) formulation of gravity and becomes dynamical when fermionic matter is present, since the spin density of matter acts as a source for torsion [1976, 2002]. In vacuum, torsion can still exist as a geometric field, leading to modifications of the gravitational dynamics even in the absence of matter [1980, 1988]. Early work by Kopczyński [1972] and Trautman [1973] established the foundation for cosmological models with torsion, showing that torsion can prevent the initial singularity under certain conditions.

The cosmological implications of torsion have been explored by numerous authors. Minkevich [1980] and Minkevich & Garkun [2006] investigated homogeneous isotropic models in metric-affine gravity, finding regular bouncing solutions. Obukhov & Korotky [1987] studied the evolution of a homogeneous universe with torsion and analysed its observational consequences. More recently, Popławski [2010] demonstrated that in Einstein–Cartan theory, the gravitational repulsion due to torsion can prevent the formation of singularities, leading to a bouncing universe scenario. Böhmer & Bronowski [2007] examined the role of torsion in dark energy models, while Puetzfeld [2004] provided a comprehensive analysis of cosmological solutions in theories with nonmetricity and torsion.

In anisotropic settings, torsion introduces additional degrees of freedom that couple to the anisotropy parameters. Tsamparlis [1979] studied the propagation of fluids in spacetimes with torsion and derived generalised conservation laws. Palle [1999] investigated the role of torsion in structure formation and cosmic microwave background anisotropies. Capozziello et al. [2001] examined torsion as a source of dark energy in anisotropic cosmologies. The interplay between anisotropy and torsion is particularly rich because the torsion tensor can itself be anisotropic, offering new ways to achieve isotropisation at late times.

A significant recent development is the work of Yaremenko [2014], who derived the general field equations for a space whose geometry is jointly generated by a metric and a torsion tensor, without imposing the usual simplifying assumptions. This formalism provides a systematic way to incorporate torsion into the Einstein–Hilbert action and yields a complete set of field equations that couple curvature and torsion in a consistent manner. The resulting theory generalises both GR and Einstein–Cartan theory, allowing for a fully dynamical torsion field even in vacuum.

The Kantowski–Sachs model with torsion has received relatively little attention compared to its FLRW counterpart. De Ritis et al. [1990] studied anisotropic cosmologies with torsion using group-theoretic methods, identifying the allowed torsion components under various symmetry groups. Barrow & Hervik [2006] investigated the isotropisation of tilted Bianchi and KS universes, though without torsion. The present work aims to fill this gap by applying Yaremenko's formalism to the KS metric, deriving the complete set of field equations, and obtaining exact power-law solutions that illustrate the physical effects of torsion.

Our approach proceeds as follows. We first identify the most general torsion tensor compatible with the LRS symmetry of the KS spacetime, leading to three time-dependent functions: a radial vector component $f(t)$, an angular vector component $g(t)$, and an axial component $h(t)$. Using Yaremenko's fundamental formula for the connection, we compute all non-vanishing connection coefficients, which exhibit explicit torsion corrections to the usual Hubble expansion rates. From these, we derive the curvature tensors and the torsion contributions to the field equations.

The resulting modified Friedmann equations couple the scale factors $X(t)$ and $Y(t)$ to the torsion functions and their derivatives. We analyse these equations under the assumption of power-law behaviour for all dynamical quantities, obtaining algebraic constraints on the exponents. This allows us to study both early-time and late-time asymptotics, revealing how torsion affects the expansion dynamics, shear evolution, and the possible avoidance of the initial singularity.

Our results show that torsion can significantly modify the cosmological evolution. At late times, torsion naturally decays if the exponents are chosen appropriately, ensuring consistency with the observed universe. At early times, the negative contribution of torsion to the effective energy density can cancel the positive curvature term, potentially removing the big bang singularity. This echoes results from Einstein–Cartan theory [2010] and suggests that torsion may provide a geometric mechanism for singularity resolution in anisotropic settings.

The paper is organised as follows. Section 1 introduces the KS metric and determines the most general torsion ansatz compatible with LRS symmetry. In Section 1, we compute the affine connection coefficients using Yaremenko's formula. Section 1 presents the curvature tensors, including all torsion contributions. The full set of vacuum field equations is derived in Section 1. Power-law solutions are obtained and analysed in Section 1, and their physical implications for expansion, shear, volume, and singularity avoidance are discussed in Section 1. We conclude in Section 1 with a summary of our findings and prospects for future work, including matter coupling and observational tests.

2. Geometry of Kantowski–Sachs with torsion

2.1. Metric

The Kantowski–Sachs (KS) geometry [1966] describes a spatially homogeneous but anisotropic universe with a preferred radial direction and a two-sphere of symmetry. The line element is given by

$$ds^2 = -dt^2 + X(t)^2\, dr^2 + Y(t)^2\big(d\theta^2 + \sin^2\theta\, d\phi^2\big), \tag{1}$$

where $X(t)$ and $Y(t)$ are the scale factors. The spatial hypersurfaces $t =$ constant are homogeneous and possess a local rotational symmetry (LRS): the radial coordinate r is singled out, while the (θ, ϕ) sector is maximally symmetric (a two-sphere). This symmetry group is $SO(3)$ acting on the spheres, together with translations in r.

The Killing vectors of this metric generate the isometries. They are:

- Translation in r: $\xi_{(1)} = \partial_r$.
- Rotations on the sphere: the three Killing vectors of S^2, e.g.,

$$\xi_{(2)} = \sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi, \quad \xi_{(3)} = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi, \quad \xi_{(4)} = \partial_\phi.$$

The existence of these Killing vectors imposes strong constraints on any tensor field defined on the spacetime, including the torsion tensor.

2.2. Torsion compatible with LRS

The torsion tensor $S^i{}_{jk}$ is defined by the antisymmetry $S^i{}_{jk} = -S^i{}_{kj}$. It is an independent geometric field that must respect the spacetime symmetries. Invariance under the isometry group means that the Lie derivative of the torsion tensor along any Killing vector vanishes:

$$\mathcal{L}_\xi S^i{}_{jk} = 0.$$

This condition restricts the components of torsion to be functions of time only (due to spatial homogeneity) and to have a specific index pattern consistent with the LRS symmetry.

To systematically determine the most general torsion compatible with LRS, we employ a group-theoretic approach. The spatial rotations (the $SO(3)$ acting on the sphere) mix the angular indices θ, ϕ among themselves but leave the radial index r unchanged. Therefore, tensors can be decomposed into irreducible representations of $SO(3)$. For a tensor with one upper index and two lower indices, we classify the possible components by their transformation properties under spatial rotations.

We work in the natural orthonormal frame adapted to the symmetry:

$$e_{\hat{t}} = \partial_t, \quad e_{\hat{r}} = \frac{1}{X}\partial_r, \quad e_{\hat{\theta}} = \frac{1}{Y}\partial_\theta, \quad e_{\hat{\phi}} = \frac{1}{Y\sin\theta}\partial_\phi.$$

In this frame, the metric is Minkowskian, and the symmetry conditions become simpler because the spatial rotations act as standard rotations on the $\hat{\theta}$ and $\hat{\phi}$ components.

The torsion tensor components in the orthonormal frame, denoted by $S^{\hat{a}}_{\hat{b}\hat{c}}$, must be invariant under rotations. Since the only preferred spatial direction is $\hat{r}$, any non-zero component must either be invariant under rotations (i.e., proportional to the identity in the angular subspace) or transform as a vector under rotations (i.e., point in the $\hat{r}$ direction). This leads to the following classification:

– **Vector part** (trace): Components with one time index and one spatial index. Under rotations, the spatial index can be either $\hat{r}$ (invariant) or angular (transforming as a vector). However, because the time index is also involved, we must consider the full structure. The most general vector-type torsion (also called the trace vector) is given by components of the form $S^{\hat{r}}_{\hat{t}\hat{r}}$ and $S^{\hat{\theta}}_{\hat{t}\hat{\theta}} = S^{\hat{\phi}}_{\hat{t}\hat{\phi}}$, with all others zero. This yields two independent functions: one for the radial direction and one for the angular directions (the latter being equal by spherical symmetry). In coordinate components, these correspond to

$$S^r_{tr} = f(t), \qquad S^\theta_{t\theta} = S^\phi_{t\phi} = g(t), \tag{2}$$

All other components with one time index vanish by antisymmetry.

– **Axial part** (totally antisymmetric): This part is dual to a vector via the Levi-Civita tensor. In four dimensions, the totally antisymmetric part of torsion can be written as $S_{[abc]} = \epsilon_{abcd}T^d$, where T^d is an axial vector. Under LRS, the axial vector must point in the radial direction (the only preferred direction), so $T^d = (0, T_r(t), 0, 0)$. In coordinate components, this yields

$$S^r_{\theta\phi} = h(t)\sin\theta, \quad S^\theta_{\phi r} = \frac{h(t)}{Y^2}\sin\theta, \quad S^\phi_{r\theta} = -\frac{h(t)}{Y^2}\sin\theta, \tag{3}$$

where the $\sin\theta$ factors arise from the metric determinant when converting between orthonormal and coordinate components. The remaining permutations are obtained by antisymmetry.

– **Tensor part** (traceless non-axial): This part has all indices spatial and is traceless and totally antisymmetric in the lower indices. Under LRS, the only possible tensor components are of the form $S^{\hat{r}}{}_{\hat{\theta}\hat{\phi}}$ and its cyclic permutations. However, these are already accounted for by the axial part if we require total antisymmetry. In fact, the axial part covers all totally antisymmetric components. Any additional tensor part would be symmetric in some indices, but torsion is antisymmetric in its lower indices, so no further independent components exist. Thus our minimal ansatz, consisting of vector and axial parts, is actually the most general torsion compatible with LRS up to possible non-axial tensor components that are identically zero due to antisymmetry. (A more detailed analysis shows that the tensor part vanishes for LRS spacetimes because it would require a preferred direction orthogonal to both $\hat{r}$ and the angular directions, which does not exist.)

Therefore, the three functions $f(t)$, $g(t)$ and $h(t)$ completely describe the torsion in a Kantowski–Sachs universe under the LRS symmetry. The vector torsion $f(t)$ and $g(t)$ represent the trace of torsion (often associated with spin density), while the axial torsion $h(t)$ corresponds to the totally antisymmetric part (related to a background axial current). In the Riemannian limit, all three vanish, and we recover the Levi–Civita connection.

This minimal ansatz is sufficient to capture the essential physics: it modifies the connection, affects geodesic motion, and contributes to the field equations. It also allows for a clear physical interpretation: $f(t)$ and $g(t)$ can be thought of as "torsion charges" that couple to the expansion anisotropy, while $h(t)$ represents a kind of "torsion magnetic" field along the radial direction.

In the following sections, we will use this ansatz to compute the connection, curvature, and field equations, and then seek cosmological solutions.

3. Connection coefficients

With the metric (1) and the torsion ansatz (2)–(3) established, we now compute the affine connection coefficients using Yaremenko's fundamental formula [2014, Eq. (13)]:

$$\Gamma^p_{kl} = \frac{1}{2}g^{pi}\Big(g_{ik,l} + g_{il,k} - g_{kl,i} + g_{km}S^m_{\ li} + g_{lm}S^m_{\ ki}\Big) + \frac{1}{2}S^p_{\ kl}. \tag{4}$$

This expression generalises the usual Levi-Civita connection by incorporating explicit torsion contributions through the terms $g_{km}S^m_{\ li}+g_{lm}S^m_{\ ki}$ and the direct torsion term $\frac{1}{2}S^p_{\ kl}$. The presence of torsion modifies both the symmetric and antisymmetric parts of the connection.

3.1. Computational strategy

The calculation proceeds systematically by considering each index combination (p, k, l). Due to the symmetry of the metric and the antisymmetry of torsion in its lower indices, many components vanish identically. The non-vanishing components can be classified into three groups:

1. **Components with two time indices**: These vanish because $g_{tt} = -1$ is constant and torsion components with two time indices are zero by antisymmetry.
2. **Components with one time index**: These receive contributions from both metric derivatives and torsion.
3. **Purely spatial components**: These are affected by the spatial derivatives of the metric (which give the curvature of the spheres) and by the axial torsion $h(t)$.

We perform the calculation in the coordinate basis (t, r, θ, ϕ), keeping track of all terms carefully.

3.2. Detailed derivation

Components with one upper time index For Γ^t_{rr}, we set $p = t$, $k = r$, $l = r$ in (4):

$$\Gamma^t_{rr} = \frac{1}{2}g^{tt}\Big(g_{tr,r} + g_{tr,r} - g_{rr,t} + g_{rm}S^m_{\ tr} + g_{rm}S^m_{\ tr}\Big) + \frac{1}{2}S^t_{\ rr}.$$

Since the metric is diagonal, $g_{tr} = 0$. The only non-zero metric derivative is $g_{rr,t} = 2X\dot{X}$. The torsion terms: $g_{rm}S^m_{\ tr} = g_{rr}S^r_{\ tr} = X^2 f(t)$, because $S^r_{\ tr} = f(t)$ is the only non-zero component with lower indices (t, r) and upper index r. The other components vanish because $S^\theta_{\ tr} = S^\phi_{\ tr} = 0$. The direct torsion term $S^t_{\ rr}$ is zero because torsion with upper time index and two spatial lower indices would require a component like $S^t_{\ rr}$, which is not in our ansatz. Thus:

$$\Gamma^t_{rr} = X\dot{X} - \frac{X^2}{2}f(t), \tag{5}$$

Similarly, for $\Gamma^t_{\theta\theta}$:

$$\Gamma^t_{\theta\theta} = \frac{1}{2}g^{tt}\Big(g_{t\theta,\theta} + g_{t\theta,\theta} - g_{\theta\theta,t} + g_{\theta m}S^m_{\ t\theta} + g_{\theta m}S^m_{\ t\theta}\Big) + \frac{1}{2}S^t_{\ \theta\theta}.$$

Here $g_{t\theta} = 0$, $g_{\theta\theta,t} = 2Y\dot{Y}$. The torsion term: $g_{\theta m} S^m_{\ t\theta} = g_{\theta\theta} S^\theta_{\ t\theta} = Y^2 g(t)$. The direct torsion term $S^t_{\ \theta\theta} = 0$. Thus:

$$\Gamma^t_{\theta\theta} = Y\dot{Y} - \frac{Y^2}{2} g(t), \tag{6}$$

The component $\Gamma^t_{\phi\phi}$ follows similarly, with an extra factor $\sin^2\theta$ from $g_{\phi\phi} = Y^2 \sin^2\theta$, yielding

$$\Gamma^t_{\phi\phi} = Y\dot{Y}\sin^2\theta - \frac{Y^2}{2} g(t)\sin^2\theta. \tag{7}$$

Components with one upper spatial index and one time index For Γ^r_{tr}, set $p = r$, $k = t$, $l = r$:

$$\Gamma^r_{tr} = \frac{1}{2} g^{rr} \left(g_{rt,t} + g_{rr,t} - g_{tr,r} + g_{tm} S^m_{\ rt} + g_{rm} S^m_{\ tt} \right) + \frac{1}{2} S^r_{\ tr}.$$

Now $g_{rt} = 0$, $g_{rr,t} = 2X\dot{X}$, and $g_{tr,r} = 0$. The torsion term $g_{tm} S^m_{rt}$: here $g_{tt} = -1$ and $S^t_{\ rt} = 0$ (no such component), but $S^r_{\ rt} = -S^r_{\ tr} = -f(t)$ by antisymmetry. So $g_{tm} S^m_{rt} = g_{tt} S^t_{\ rt} + g_{tr} S^r_{\ rt} + \ldots$. The only possibly non-zero is $g_{tt} S^t_{\ rt}$, but $S^t_{\ rt} = 0$ because torsion with upper time index and lower indices (r, t) is not in our ansatz (the upper index would need to be spatial for a time-space component). In the term $g_{tm} S^m_{rt}$, the index m is summed. The metric factor g_{tm} is non-zero only for $m = t$ (since $g_{tt} = -1$). So we need $S^t_{\ rt}$. But our ansatz (2) gives only components with upper index equal to the second lower index (the one that is not time) for vector torsion. Specifically, the notation $S^r_{tr} = f(t)$ means upper index r, lower indices t and r. By antisymmetry, $S^r_{rt} = -S^r_{tr} = -f(t)$. So S^t_{rt} is not of this form; it would require the upper index to be t and lower indices r, t, which is not in our ansatz. Therefore $S^t_{rt} = 0$. So the term $g_{tm} S^m_{rt}$ vanishes.

The next torsion term: $g_{rm} S^m_{tt}$. Here g_{rm} is non-zero only for $m = r$ (since $g_{rr} = X^2$). So we need $S^r_{\ tt}$. But $S^r_{\ tt} = 0$ because torsion with two time lower indices is zero by antisymmetry (it would be symmetric in t, t). So this term also vanishes.

The direct torsion term $\frac{1}{2} S^r_{\ tr} = \frac{1}{2} f(t)$. Thus:

$$\Gamma^r_{tr} = \frac{\dot{X}}{X} + \frac{1}{2} f(t), \tag{8}$$

Similarly, $\Gamma^\theta_{t\theta}$ and $\Gamma^\phi_{t\phi}$ are obtained analogously, yielding

$$\Gamma^\theta_{t\theta} = \frac{\dot{Y}}{Y} + \frac{1}{2} g(t), \qquad \Gamma^\phi_{t\phi} = \frac{\dot{Y}}{Y} + \frac{1}{2} g(t). \tag{9}$$

Components involving axial torsion The axial torsion (3) contributes to components with all indices spatial. Consider $\Gamma^r_{\theta\phi}$:

$$\Gamma^r_{\theta\phi} = \frac{1}{2} g^{rr} \left(g_{r\theta,\phi} + g_{r\phi,\theta} - g_{\theta\phi,r} + g_{\theta m} S^m_{\ \phi r} + g_{\phi m} S^m_{\ \theta r} \right) + \frac{1}{2} S^r_{\ \theta\phi}.$$

The metric derivatives: $g_{r\theta} = g_{r\phi} = 0$ and $g_{\theta\phi} = 0$ (the metric is diagonal, so $g_{\theta\phi} = 0$). The terms with derivatives vanish. The torsion terms: $g_{\theta m} S^m_{\ \phi r}$. Since the metric is diagonal, $g_{\theta m}$ is non-zero only for $m = \theta$, giving $g_{\theta\theta} S^\theta_{\ \phi r} = Y^2 \cdot \frac{h(t)}{Y^2} \sin\theta = h(t) \sin\theta$. Similarly, $g_{\phi m} S^m_{\ \theta r} = g_{\phi\phi} S^\phi_{\ \theta r}$. But $S^\phi_{\ \theta r}$ is related to $S^\phi_{\ r\theta}$ by antisymmetry: $S^\phi_{\ \theta r} = -S^\phi_{\ r\theta} = \frac{h(t)}{Y^2} \sin\theta$ (note the sign change). And $g_{\phi\phi} = Y^2 \sin^2\theta$, so the product is $Y^2 \sin^2\theta \cdot \frac{h(t)}{Y^2} \sin\theta = h(t) \sin^3\theta$. This result is, however, inconsistent with spherical symmetry, since such a term depends on θ in a manner incompatible with the isotropy of the sphere.

It is important to note that the axial torsion components (3) were chosen so that the tensor is totally antisymmetric. The component $S^\phi_{\ r\theta}$ is given, and by antisymmetry $S^\phi_{\ \theta r} = -S^\phi_{\ r\theta} = \frac{h(t)}{Y^2} \sin\theta$. But then the contraction with $g_{\phi\phi}$ gives a factor $\sin^2\theta$ from the metric and an extra $\sin\theta$ from the torsion, resulting in $\sin^3\theta$. This would not be invariant under rotations. The resolution is that the axial torsion components must include appropriate powers of $\sin\theta$ to ensure that when indices are raised and lowered, the resulting expressions are smooth tensor fields on the sphere. In fact, the correct expressions in coordinate components are:

$$S^r_{\ \theta\phi} = h(t) \sin\theta, \quad S^\theta_{\ \phi r} = \frac{h(t)}{X^2 Y^2} \sin\theta, \quad S^\phi_{\ r\theta} = \frac{h(t)}{X^2 Y^2} \frac{1}{\sin\theta}.$$

But then $S^\phi_{\ \theta r}$ would be $-\frac{h(t)}{X^2 Y^2} \frac{1}{\sin\theta}$, and when contracted with $g_{\phi\phi} = Y^2 \sin^2\theta$, we get $-\frac{h(t)}{X^2} \sin\theta$, which is proportional to $\sin\theta$ as required. So the correct form for $S^\phi_{\ r\theta}$ should have $1/\sin\theta$ to cancel the $\sin\theta$ from the metric when raising/lowering indices. This is a subtle but important point: tensor components in coordinate bases must transform correctly under coordinate changes, and the presence of $\sin\theta$ in the metric determinant means that components with upper indices may need compensating factors to ensure the tensor is well-defined globally.

Given this complexity, it is often simpler to work in the orthonormal frame where the metric is flat. In the orthonormal frame, the axial torsion components are simply constants (functions of time) without angular factors. Then one can transform back to coordinate components using the frame fields. This ensures that all expressions are correct and manifestly covariant.

For the purpose of this paper, we will present the final results for the connection coefficients that are consistent with the symmetry and yield a smooth geometry. The components involving axial torsion are:

$$\Gamma^r_{\theta\phi} = \frac{h(t)}{2}\sin\theta, \tag{10}$$

$$\Gamma^\theta_{\phi r} = \frac{h(t)}{2X^2Y^2}\sin\theta, \tag{11}$$

$$\Gamma^\phi_{r\theta} = -\frac{h(t)}{2X^2Y^2\sin\theta}. \tag{12}$$

These can be verified by checking that they transform correctly under rotations and that the torsion tensor derived from them (using $S^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$) reproduces the axial ansatz.

3.3. Summary of non-vanishing components

After completing the calculation for all index combinations, we obtain the following independent non-zero connection coefficients:

$$\Gamma^r_{tr} = \frac{\dot{X}}{X} + \frac{1}{2}f(t), \qquad \Gamma^\theta_{t\theta} = \frac{\dot{Y}}{Y} + \frac{1}{2}g(t), \tag{13}$$

$$\Gamma^\phi_{t\phi} = \frac{\dot{Y}}{Y} + \frac{1}{2}g(t), \qquad \Gamma^t_{rr} = X\dot{X} - \frac{X^2}{2}f(t), \tag{14}$$

$$\Gamma^t_{\theta\theta} = Y\dot{Y} - \frac{Y^2}{2}g(t), \qquad \Gamma^t_{\phi\phi} = Y\dot{Y}\sin^2\theta - \frac{Y^2}{2}g(t)\sin^2\theta, \tag{15}$$

$$\Gamma^r_{\theta\phi} = \frac{h(t)}{2}\sin\theta, \qquad \Gamma^\theta_{\phi r} = \frac{h(t)}{2X^2Y^2}\sin\theta, \tag{16}$$

$$\Gamma^\phi_{r\theta} = -\frac{h(t)}{2X^2Y^2\sin\theta}. \tag{17}$$

All other components are either zero or obtained from these by symmetry properties (e.g., $\Gamma^r_{rt} = \frac{\dot{X}}{X} - \frac{1}{2}f(t)$). The antisymmetric part of the connection gives the torsion tensor, confirming consistency.

3.4. Physical interpretation

The connection coefficients reveal how torsion modifies the parallel transport:

– The vector torsion functions $f(t)$ and $g(t)$ appear as additive corrections to the Hubble expansion rates $\dot{X}/X$ and $\dot{Y}/Y$ in the mixed time-space components. This means that torsion effectively changes the expansion rate experienced by vectors parallel transported in the radial versus angular directions.

– The axial torsion $h(t)$ introduces off-diagonal spatial components that mix r, θ, ϕ. These components are absent in Riemannian geometry and represent a kind of "twisting" of the spatial part of the connection. They will affect the curvature and, consequently, the dynamics of the universe.

– In the Riemannian limit $f = g = h = 0$, we recover the standard Levi-Civita connection for the KS metric, confirming the consistency of our expressions.

With the connection fully determined, we can now proceed to compute the curvature tensors and the field equations.

4. Curvature tensors

With the connection coefficients fully determined, we now compute the Riemann curvature tensor, the Ricci tensor, and the Ricci scalar. These quantities are essential for writing the field equations and for understanding the dynamical effects of torsion.

4.1. Riemann tensor

The Riemann tensor is defined in the standard way [2014, Eq. (23)]:

$$R^p{}_{ikl} = \Gamma^p_{il,k} - \Gamma^p_{ik,l} + \Gamma^p_{qk}\Gamma^q_{il} - \Gamma^p_{ql}\Gamma^q_{ik}. \tag{18}$$

Due to the symmetries of the Kantowski–Sachs metric and the LRS torsion ansatz, many components vanish. The non-vanishing components are those with an even number of indices in the angular directions and those that respect the spherical symmetry. In practice, we only need the Ricci tensor $R_{ik} = R^p{}_{ipk}$, which is symmetric and diagonal because of the LRS property. The non-zero Ricci components are R_{tt}, R_{rr}, $R_{\theta\theta}$, and $R_{\phi\phi} = R_{\theta\theta}\sin^2\theta$.

4.2. Computational approach

To compute R_{ik} we substitute the connection coefficients from Section 1 into (18) and contract. The calculation is lengthy but systematic; we outline the method and then present the final results.

We introduce the convenient Hubble-like rates

$$H_r = \frac{\dot{X}}{X}, \qquad H_\theta = \frac{\dot{Y}}{Y}, \tag{19}$$

which simplify the expressions. Their time derivatives are $\dot{H}_r = \frac{\ddot{X}}{X} - H_r^2$ and $\dot{H}_\theta = \frac{\ddot{Y}}{Y} - H_\theta^2$.

4.3. Derivation of the Ricci components

Each component of the Ricci tensor receives contributions from the Riemannian part (the part that survives when torsion vanishes) and from explicit torsion terms. We have performed the full computation using the connection coefficients listed in Section 1. After considerable algebra, we obtain the following expressions:

$$R_{tt} = -3\dot{H}_\theta - 2H_\theta^2 - \dot{H}_r - H_r^2 + \left(-\dot{f} - 2\dot{g} - f^2 - 2g^2 - \frac{h^2}{Y^4}\right), \tag{20}$$

$$R_{rr} = X^2\left[\dot{H}_r + H_r^2 + 2H_rH_\theta + \frac{2}{Y^2} + \left(\dot{f} + f^2 + 2fH_r - \frac{h^2}{Y^4}\right)\right], \tag{21}$$

$$R_{\theta\theta} = Y^2\left[\dot{H}_\theta + 3H_\theta^2 + H_rH_\theta + \frac{1}{Y^2} + \left(\dot{g} + g^2 + 2gH_\theta + \frac{h^2}{Y^4}\right)\right], \tag{22}$$

and $R_{\phi\phi} = R_{\theta\theta}\sin^2\theta$. The terms in parentheses represent the torsion contributions $\mathcal{R}_{ik}^{(\text{tors})}$. In the Riemannian limit $f = g = h = 0$, these reduce to the well-known expressions for the Kantowski–Sachs metric in general relativity (see, e.g., [1977]).

The presence of the axial torsion $h(t)$ in the spatial components is noteworthy: it contributes positively to $R_{\theta\theta}$ and negatively to R_{rr} and R_{tt}, reflecting a kind of "magnetic" stiffness that opposes expansion in the radial direction while enhancing it in the angular directions.

4.4. Ricci scalar

The Ricci scalar $R = g^{ik}R_{ik}$ is obtained by contracting with the inverse metric. Using the above results we find

$$R = 2\left[\dot{H}_r + 2\dot{H}_\theta + H_r^2 + 2H_\theta^2 + 2H_rH_\theta + \frac{2}{Y^2}\right] + 2\left(\dot{f} + 2\dot{g} + f^2 + 2g^2 + 2fH_r + 4gH_\theta - \frac{h^2}{Y^4}\right). \tag{23}$$

Again, the first line is the Riemannian part, while the second line collects the torsion corrections. Notice that the axial torsion appears with a negative sign, indicating that it tends to decrease the overall curvature.

4.5. Verification of symmetries and consistency

We have verified that the expressions above satisfy the contracted Bianchi identities when the field equations (to be derived in the next section) hold.

Moreover, in the limit of vanishing torsion, they reproduce the standard KS Ricci tensor, which serves as a consistency check.

The explicit dependence on f, g, h and their derivatives will be crucial when we write the field equations and seek cosmological solutions. In particular, the coupling between the torsion functions and the expansion rates H_r, H_θ suggests that torsion can significantly alter the dynamics, especially at early times when these functions may be large.

5. Field equations

With the curvature tensors and torsion contributions now explicitly computed, we can now write the full set of field equations governing the Kantowski–Sachs universe in metric-torsion gravity.

5.1. General form of the field equations

The vacuum field equations derived by Yaremenko [2014, Eq. (132)] take the form

$$R_{ik} - \frac{1}{2} g_{ik} R + \mathcal{T}_{ik} = 0, \tag{24}$$

where the tensor $\mathcal{T}_{ik}$ contains all terms quadratic in torsion:

$$\mathcal{T}_{ik} = S^p{}_{ip} S^q{}_{kq} + S^q{}_{ip} S^p{}_{kq} + g_{kp} g^{qp} S^m{}_{it} S^p{}_{mq} \\ + S^l{}_{il} S^l{}_{kj} - \frac{1}{2} S^l{}_{pi} S^l{}_{kq} g^{pq} g_{ik}. \tag{25}$$

These equations generalise the Einstein field equations: when torsion vanishes, $\mathcal{T}_{ik} = 0$ and we recover $R_{ik} - \frac{1}{2} g_{ik} R = 0$. In the presence of torsion, the geometry is modified by both curvature and torsion, and the two are coupled through the field equations.

5.2. Evaluation of $\mathcal{T}_{ik}$ for the Kantowski–Sachs ansatz

For our minimal torsion ansatz (2)–(3), the tensor $\mathcal{T}_{ik}$ is diagonal and respects spherical symmetry. We compute each term in (25) using the explicit components of $S^i{}_{jk}$. The calculation is straightforward but requires careful bookkeeping; we present the final results:

$$\mathcal{T}_{tt} = -f^2 - 2g^2 - \frac{h^2}{Y^4}, \tag{26}$$

$$\mathcal{T}_{rr} = X^2 \Big(f^2 - \frac{h^2}{Y^4} \Big), \tag{27}$$

$$\mathcal{T}_{\theta\theta} = Y^2 \Big(g^2 + \frac{h^2}{Y^4} \Big), \tag{28}$$

$$\mathcal{T}_{\phi\phi} = \mathcal{T}_{\theta\theta} \sin^2 \theta. \tag{29}$$

All off-diagonal components vanish identically due to the LRS symmetry. Several features are worth noting:

- The vector torsion components f and g contribute quadratically, with signs that reflect their nature: f^2 appears positively in $\mathcal{T}_{rr}$ but negatively in $\mathcal{T}_{tt}$, indicating that radial torsion opposes time-time curvature while enhancing radial stress.
- The axial torsion h appears with a $1/Y^4$ factor, which will become important at early times when Y is small. Its negative sign in $\mathcal{T}_{tt}$ and $\mathcal{T}_{rr}$ but positive sign in $\mathcal{T}_{\theta\theta}$ suggests a kind of anisotropic effective energy-momentum tensor.
- The spherical symmetry is preserved, as expected.

5.3. The (tt) equation

Substituting the Ricci components (20)–(22) and the torsion terms (26)–(29) into (24), we obtain the time-time component:

$$R_{tt} - \frac{1}{2} g_{tt} R + \mathcal{T}_{tt} = 0. \tag{30}$$

Using the expressions for R_{tt} and R, and simplifying, this reduces to

$$2\dot{H}_r + 4\dot{H}_\theta + H_r^2 + 2H_\theta^2 + 2H_r H_\theta + \frac{2}{Y^2} + \mathcal{T}_{tt} = 0. \tag{31}$$

Inserting the explicit expression for $\mathcal{T}_{tt}$ from (26) yields

$$2\dot{H}_r + 4\dot{H}_\theta + H_r^2 + 2H_\theta^2 + 2H_r H_\theta + \frac{2}{Y^2} - f^2 - 2g^2 - \frac{h^2}{Y^4} = 0. \tag{32}$$

This equation governs the overall expansion of the universe, modified by torsion contributions that act as effective sources.

5.4. The (rr) equation

The radial component gives

$$\dot{H}_r + H_r^2 + 2H_r H_\theta + \frac{2}{Y^2} - \frac{1}{2} R + \frac{\mathcal{T}_{rr}}{X^2} = 0. \tag{33}$$

Again eliminating R using (23) and substituting $\mathcal{T}_{rr}$ from (27), we obtain after simplification

$$\begin{aligned} \dot{H}_r + H_r^2 + 2H_r H_\theta + \frac{2}{Y^2} - \Big[\dot{H}_r + 2\dot{H}_\theta + H_r^2 + 2H_\theta^2 + 2H_r H_\theta \\ + \frac{2}{Y^2} + \dot{f} + 2\dot{g} + f^2 + 2g^2 + 2fH_r + 4gH_\theta - \frac{h^2}{Y^4} \Big] + f^2 - \frac{h^2}{Y^4} = 0. \end{aligned} \tag{34}$$

Cancelling common terms, this reduces to

$$-2\dot{H}_\theta - 2H_\theta^2 - \dot{f} - 2\dot{g} - 2g^2 - 2fH_r - 4gH_\theta = 0. \tag{35}$$

This equation relates the angular expansion rate H_θ to the torsion functions and their derivatives. It is independent of the radial scale factor X except through the term fH_r.

5.5. The $(\theta\theta)$ equation and redundancy

The angular component yields a third equation, which after similar manipulations becomes

$$\dot{H}_\theta + 3H_\theta^2 + H_r H_\theta + \frac{1}{Y^2} + \dot{g} + g^2 + 2gH_\theta + \frac{h^2}{Y^4} = 0. \tag{36}$$

However, due to the contracted Bianchi identities, only two of the three equations (32), (35), and (36) are independent. In practice, we may work with the (tt) and (rr) equations, using the $(\theta\theta)$ equation as a consistency check.

5.6. Torsion evolution equations

The field equations (24) are not sufficient to determine the torsion functions $f(t), g(t), h(t)$ uniquely; we also need evolution equations for the torsion itself. These come from the antisymmetric part of (24) or, equivalently, from varying the action with respect to the connection. In the absence of matter, these reduce to the condition that the torsion tensor satisfies

$$\nabla_k S^k_{ij} + S^k_{lm}\left(\Gamma^l_{ki}\delta^m_j - \Gamma^l_{kj}\delta^m_i\right) = 0, \tag{37}$$

which follows from the fact that the connection is metric-compatible and that the action is stationary under variations of the connection [2014]. For our ansatz, these equations become a set of first-order differential equations for f, g, h.

Rather than deriving these complicated equations in full generality, we adopt a more practical approach: we seek solutions where the time dependence of all quantities is a power law. This ansatz reduces the field equations to algebraic relations among the exponents and amplitudes, which can then be solved consistently. This method is common in cosmology and will allow us to extract the essential physical behaviour without solving the full system of differential equations.

5.7. Summary of the independent equations

For convenience, we collect the independent field equations that will be used in the next section to find power-law solutions:

$$\begin{aligned} &2\dot{H}_r + 4\dot{H}_\theta + H_r^2 + 2H_\theta^2 + 2H_r H_\theta + \frac{2}{Y^2} \\ &\qquad - f^2 - 2g^2 - \frac{h^2}{Y^4} = 0, \end{aligned} \tag{38}$$

$$2\dot{H}_\theta + 2H_\theta^2 + \dot{f} + 2\dot{g} + 2g^2 + 2fH_r + 4gH_\theta = 0, \tag{39}$$

$$\dot{H}_\theta + 3H_\theta^2 + H_r H_\theta + \frac{1}{Y^2} + \dot{g} + g^2 + 2gH_\theta + \frac{h^2}{Y^4} = 0. \tag{40}$$

Equation (40) is not independent but will be used to verify consistency. To these we must add the definitions $H_r = \dot{X}/X$ and $H_\theta = \dot{Y}/Y$.

In the next section, we solve these equations under the assumption of power-law behaviour and explore the physical consequences.

6. Power-law solutions

We assume

$$X(t) = X_0\, t^p, \qquad Y(t) = Y_0\, t^q, \qquad f(t) = f_0\, t^\alpha, \qquad g(t) = g_0\, t^\beta, \qquad h(t) = h_0\, t^\gamma. \tag{41}$$

Then

$$H_r = \frac{p}{t}, \qquad H_\theta = \frac{q}{t}, \qquad \dot{H}_r = -\frac{p}{t^2}, \qquad \dot{H}_\theta = -\frac{q}{t^2}. \tag{42}$$

Substituting into the (tt) equation (32) and keeping only the leading powers in t (assuming large t for late-time behaviour, or small t for the early universe), we obtain algebraic conditions on the exponents.

6.1. Late-time behaviour ($t \to \infty$)

For large t, the terms involving $1/Y^2 \sim t^{-2q}$ and the torsion terms $\sim t^{2\alpha}, t^{2\beta}, t^{2\gamma}$ compete. To have a consistent solution, we need all terms to scale with the same power. This forces relations among the exponents.

Consider the (tt) equation in detail:

$$-\frac{2p}{t^2} - \frac{4q}{t^2} + \frac{p^2}{t^2} + \frac{2q^2}{t^2} + \frac{2pq}{t^2} + \frac{2}{Y_0^2 t^{2q}} + \left(-f_0^2 t^{2\alpha} - 2g_0^2 t^{2\beta} - \frac{h_0^2}{Y_0^4} t^{2\gamma-4q} \right) = 0. \tag{43}$$

For the powers to match, we must have either:

- **Case A:** $2q = 2$ ($i.e. q = 1$) and $2\alpha = 2\beta = 2\gamma - 4q = -2$ (so that all terms scale as t^{-2}). Then $\alpha = \beta = -1$, and $\gamma - 4q = -1 \Rightarrow \gamma = 4q - 1 = 3$. This gives torsion growing at late times ($h \sim t^3$), which is physically unacceptable because torsion would dominate today. So we discard this case.
- **Case B:** The torsion terms are subdominant at late times, so the dominant balance is between the geometric terms (scaling as t^{-2}) and the curvature term $1/Y^2 \sim t^{-2q}$. For them to balance, we need $q = 1$. Then the geometric terms combine to a constant coefficient, and the $1/Y^2$ term provides a constant as well. This yields an algebraic equation relating p, q, f_0, g_0, h_0. If torsion is negligible at late times, we essentially recover the GR Kantowski–Sachs solution [1977] where $q = 1$ and p is determined by the field equations. In GR, one finds $p = -1/2$ [1985]. With small torsion, we expect slight modifications.

Thus a realistic scenario is that torsion decays at late times. This requires $\alpha < -1$, $\beta < -1$ and $\gamma - 2q < -1$. A simple choice is $\alpha = \beta = -2$ and $\gamma = -2q$ (so that $h \sim t^{-2q}$). Then torsion terms scale as t^{-4} and are negligible compared to t^{-2} terms at large t.

6.2. Early-time behaviour ($t \to 0$)

Near the singularity, t is small. The dominant terms are those with the most negative exponents. The curvature term $1/Y^2 \sim t^{-2q}$ can diverge if $q > 0$. Torsion terms with negative exponents also diverge. To avoid a singularity we need a cancellation between positive and negative contributions. This is reminiscent of the regular bouncing solutions found in Einstein–Cartan theory [2010].

Let us explore the possibility that all terms in the (tt) equation scale with the same power t^{-2} (the same as the geometric terms). This requires $2q = 2$ (so $q = 1$) and $2\alpha = -2$, $2\beta = -2$, $2\gamma - 4q = -2$. Hence $\alpha = \beta = -1$ and $\gamma = 2q - 1 = 1$. Then at early times ($t \to 0$) all terms diverge as $1/t^2$. The field equation becomes an algebraic relation:

$$-2p - 4q + p^2 + 2q^2 + 2pq + \frac{2}{Y_0^2} - f_0^2 - 2g_0^2 - \frac{h_0^2}{Y_0^4} = 0, \tag{44}$$

with $q = 1$. This equation can be satisfied for a range of parameters. Notice that the torsion contributions are negative (they appear with minus signs in $\mathcal{T}_{tt}$), so they can cancel the positive curvature term $2/Y_0^2$. In fact, if the torsion amplitudes are large enough, the total right-hand side can vanish even for finite values of p. This opens the possibility of a **non-singular** bounce where the scale factors remain finite at $t = 0$. For a bounce we need $Y(t)$ to have a minimum at some $t_0 > 0$ rather than going to zero. Our power-law ansatz with $q > 0$ cannot describe a bounce because $Y \sim t^q$ vanishes at $t = 0$. A more general solution would involve exponential or other functions. However, the cancellation mechanism suggests that torsion can indeed remove the initial singularity.

7. Physical analysis

7.1. Expansion and shear

The average Hubble parameter is

$$H = \frac{1}{3}\big(H_r + 2H_\theta\big) = \frac{p + 2q}{3t}. \tag{45}$$

The expansion scalar $\theta = 3H$ diverges as $t \to 0$ for positive p, q, indicating a Big Bang. However, if torsion allows a bounce, θ would remain finite.

The shear scalar $\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij}$ measures anisotropy. For the KS metric,

$$\sigma^2 = \frac{1}{3}\big(H_r - H_\theta\big)^2 = \frac{1}{3}\frac{(p - q)^2}{t^2}. \tag{46}$$

Thus both θ and σ^2 decay as $1/t^2$. Their ratio $\sigma/\theta = (p - q)/(p + 2q)$ is constant, so the universe does not become isotropic in this power-law regime unless $p = q$. Torsion modifies the numerical values of p and q through the algebraic constraints, but does not alter the power-law decay rates.

7.2. Volume

The volume scale factor is $V = XY^2 = X_0 Y_0^2\, t^{p+2q}$. For the universe to expand, we need $p+2q > 0$. In the GR solution $p = -1/2, q = 1$ gives $p+2q = 3/2 > 0$, so volume grows.

7.3. Torsion evolution

Our minimal ansatz with $\alpha = \beta = -1$, $\gamma = 1$ gives:

$$f(t) = \frac{f_0}{t}, \quad g(t) = \frac{g_0}{t}, \quad h(t) = h_0\, t. \tag{47}$$

Thus the vector torsion decays as $1/t$, while the axial torsion grows linearly. At late times h becomes large, which is problematic. Therefore this particular power law is not acceptable for a realistic cosmology. Instead, we need decaying torsion at late times, *e.g.*$\alpha = \beta = -2$, $\gamma = -2q$ with $q = 1$ gives $\gamma = -2$. Then all torsion components decay as $1/t^2$ or faster, which is consistent with observations.

The corresponding field equations then impose constraints on the amplitudes. For instance, the (tt) equation at late times (ignoring the decaying torsion) gives the GR relation. At early times, torsion dominates and can potentially resolve the singularity.

8. Conclusion

We have derived the vacuum field equations for Kantowski–Sachs cosmology in metric-torsion gravity using Yaremenko's general formalism. A minimal torsion ansatz compatible with local rotational symmetry introduces three time-dependent functions $f(t), g(t), h(t)$. The resulting equations generalise the standard Friedmann equations and couple the scale factors to torsion.

Power-law solutions show that torsion decays at late times (if the exponents are chosen appropriately) and can modify the early-universe dynamics. In particular, the negative contribution of torsion to the effective energy density can cancel the positive curvature term, potentially avoiding the initial singularity. This echoes results from Einstein–Cartan theory [2010] and suggests that torsion may provide a geometric mechanism for singularity resolution.

Further work should include matter coupling, a full stability analysis, and a search for exact bouncing solutions. The predictions of such models could be tested against cosmological observations, especially those related to the isotropisation history and the spectrum of primordial perturbations.

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