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Scanning of stellar images by a slit diafragm and estimation of their parameters. I. Approximation of turbulent disks by a Gaussian distribution

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Abstract

Exact analytical expressions are derived for evaluation of the smoothed sizes of the stellar images when they are scanned by an infinitely long slit with a finite width $2h$. Two cases are considered: (i) the data are approximated by a Gaussian curve by the least squares method; (ii) the data are approximated by a Gaussian curve, which has the same y-coordinate of the center of weight as the smoothed curve.

Key words: stellar images, scanning.

I. Introduction

It is well known that for a long enough exposure times (e.g., to smear out the speckle structure) the distribution of light intensity within the turbulent disks of stellar images may be well fitted by a Gaussian function with a dispersion σ^2. This statement is true if the observations are made by a ground-based telescope with a large aperture, greater than $\sim$ 50 cm, and if also a narrow transmission band filters are used to eliminate the atmospheric dispersion at large zenith angles. The conclusions following below are, obviously, applicable when photographic stellar images are scanned by a long slit window with a finite width $2h$.

Let us introduce in the focal plane of the telescope a rectangular coordinate system (x,y) with an origin at the center of the turbulent stellar disk. Then the addopted unsmoothed intensity distribution is given by the expression

$$(1) \quad g(x,y)=\frac{S}{2\pi\sigma_0^2}\exp\left(-\frac{x^2+y^2}{2\sigma_0^2}\right) ,$$

where S is a normalization constant and σ_0^2 is the dispersion of the two-dimensional Gaussian distribution. Addopting a circular symmetry of the stellar images we set $\sigma_x=\sigma_y=\sigma_0$. The quantity σ_0 is connected with the Fried's parameter r_0 , giving the correlation radius of the atmospheric turbulence (Fried, 1966); the greater r_0 , the smaller is σ_0 . Because we are interesting only of the relative light distribution in the stellar images and not of their total photometric fluxes we set further $S = 1$. For simplicity we set also $\sigma_0 = 1$, remembering hereafter in this paper that all linear scales are measured in unit of length σ_0 .

II. Definition of the problem

We shall describe the rectangular window of the scanning diafragm by an infinitely long slit, with a finite width $2h$. The slit is alighned along the axis y . Denoting by x the abcissa of the midlle line of the slit, the intensity flux $2h\tilde{g}(x)$ transmitted by the diafragm is given as follows

$$(2) \quad \tilde{g}(x)=\int_{x-h}^{x+h}\left[\int_{-\infty}^{\infty}g(x,y)\,dy\right]dx=(2h)^{-1}\left[\Phi(x+h)-\Phi(x-h)\right],$$

where $\Phi(x)=(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-t^2/2)\,dt$ is the error integral (Korn et al., 1977; Janke et al., 1977) and performing the integration by y in (2) we have get the multiplier $\sqrt{2\pi}$. The smoothed distribution $\tilde{g}(x)$ is not exactly a Gaussian curve but we may approximate it by a Gaussian one with a dispersion σ^2

$$\tilde{g}^G(x) = \left(\sqrt{2\pi}\,\sigma\right)^{-1}\exp\left(-x^2/2\sigma^2\right) \quad (3)$$

Of course, if the width of the slit $2h$ is not too large, we would expect that the value of σ is not very different from that of $\sigma_0 = 1$ of the unsmoothed distribution $g(x,y)$. In this paper we give exact functional relations between σ, σ_0 and h for two different cases: (i) the approximation (3) is performed by the least squares method; and (ii) the approximation (3) is obtained by equating the centres of weight (by the y-coordinates) of the areas constrained by the unsmoothed and smoothed curves, respectively. These results may be used to correct the measured σ's for the finite width of the slit $2h$, when very precize values of σ_0's are needed to be known. Obviously, the results in this paper are applicable not only in the case of a real-time astronomical scanning of stellar images. They are useful in the slit microphotometry of photographic stellar images.

III. Approximation of $\tilde{g}(x)$ with a Gaussian curve by the least squares method

In this section we shall denote $\tilde{g}^G$ from (3) by $\tilde{g}_{LS}$.

The approximation by the method of least squares in our case is described by the condition (Brandt, 1975)

$$(4)\quad \int_{-\infty}^{\infty}\left[\tilde{g}(x)-\tilde{g}_{LS}(x)\right]^2 dx = \min ,$$

or equivalently

$$(5)\quad \int_{-\infty}^{\infty}\left[\tilde{g}(x)-\tilde{g}_{LS}(x)\right]\frac{d}{d\sigma}\tilde{g}_{LS}(x)\,dx = 0.$$

This is an equation for the desired quantity σ as a function of $2h$ and σ_o (in units $\sigma_o = 1$). Substituting (2) and (3) in (5), the least squares condition for σ becomes

$$(6)\quad \int_{-\infty}^{\infty}\left[\Phi(x+h)-\Phi(x-h)-\frac{2h}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)\right]$$

$$\times\ (x^2-\sigma^2)\exp\left(-\frac{x^2}{2\sigma^2}\right)dx = 0.$$

Further we denote by $\mathcal{K}_o(h)$ the integral

$$(7)\quad \mathcal{K}_o(h) = \int_{-\infty}^{\infty}\Phi(x+h)\exp(-x^2/2\sigma^2)\,dx ,$$

and rewrite (6) as a sum of six integrals

$$(8)\quad \sum_{i=1}^{6}\mathcal{K}_i(h) = 0 ,$$

where $\mathcal{K}_i(h)$ $(i = 1,\ldots,6)$ are defined further in this section as follows

$$(9)\quad \mathcal{K}_1(h) = \int_{-\infty}^{\infty}\Phi(x+h)\,x^2\exp(-x^2/2\sigma^2)\,dx .$$

Integrating by parts we obtain that

$$(10)\quad \mathcal{K}_1(h) = \sigma^2 \mathcal{K}_o(h) + \frac{\sigma^2}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} x \exp\left[-\left(\frac{1+\sigma^2}{2\sigma^2}\right) x^2 + 2\left(-\frac{h}{2}\right) x \right] dx \right\} \exp\left(-\frac{h^2}{2}\right).$$

Gradshtein et al.(1963) (formula 3.462.2) have evaluated the integral

$$(11)\quad \int_{-\infty}^{\infty} x^n \exp(-px^2 + 2qx)\, dx = \frac{1}{2^{n-1}p}\sqrt{\frac{\pi}{p}}\, \frac{d^{n-1}}{dq^{n-1}} \left[q \exp\left(\frac{q^2}{p}\right) \right]$$

for n positive integer and $p > 0$. In our case for $n = 1$, $p = (1+\sigma^2)/2\sigma^2$ and $q = -h/2$ we get finally, according to (10)

$$(12)\quad \mathcal{K}_1(h) = \sigma^2 \mathcal{K}_o(h) - \frac{\sigma^5 h}{(1+\sigma^2)^{3/2}} \exp\left[-\frac{h^2}{2(1+\sigma^2)} \right],$$

$$(13)\quad \mathcal{K}_2(h) = -\sigma^2 \int_{-\infty}^{\infty} \Phi(x+h) \exp(-x^2/2\sigma^2)\, dx = -\sigma^2 \mathcal{K}_o(h),$$

$$(14)\quad \mathcal{K}_3(h) = -\int_{-\infty}^{\infty} \Phi(x-h)\, x^2 \exp(-x^2/2\sigma^2)\, dx = -\mathcal{K}_1(-h),$$

$$(15)\quad \mathcal{K}_4(h) = \sigma^2 \int_{-\infty}^{\infty} \Phi(x-h) \exp(-x^2/2\sigma^2)\, dx = \sigma^2 \mathcal{K}_o(-h),$$

and

$$(16)\quad \mathcal{K}_5(h) = -\frac{2h}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} x^2 \exp(-x^2/\sigma^2)\,dx\,.$$

Integrating by parts and using the result that the error integral at infinity is equal to unity (e.g., $\Phi(\infty) = 1$), we obtain

$$(17)\quad \mathcal{K}_5(h) = -h\sigma^2/\sqrt{2}\,.$$

Finally we get

$$(18)\quad \mathcal{K}_6(h) = \frac{2h\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-x^2/\sigma^2)\,dx = 2h\sigma^2/\sqrt{2}\,.$$

Substituting the expressions (12) - (15),(17) and (18) into the least squares condition (8),and performing the summation we obtain the desired functional dependence between the slit width $2h$ and the dispersion σ^2 of the Gaussian distribution by which the smoothed curve $\tilde{g}(x)$ from (2) has been approximated.Note that the integrals $\mathcal{K}_0(h)$ and $\mathcal{K}_0(-h)$ during summing up annihilate,so there is not need to compute them.

$$(19)\quad \frac{\sigma^3}{(1+\sigma^2)^{3/2}}\exp\left[-\frac{h^2}{2(1+\sigma^2)}\right] = \frac{1}{2\sqrt{2}}\,.$$

This relation uniquely determines the dependence between h and σ because the multipliers $\sigma^3/(1+\sigma^2)^{3/2}$ and $\exp[-h^2/2(1+\sigma^2)]$ for every one fixed value of h are strict monotonically increasing functions of σ ,and also the exponential factor strictly decreases with increasing of h

(for a fixed value of σ). It is more useful to express h as a function of σ . For this purpose we may take the natural logarithms from two sides of (19) and the desired final expression is

$$(20) \quad 2h = 2\left[(1+\sigma^2)\ln\frac{8\sigma^6}{(1+\sigma^2)^3}\right]^{1/2} .$$

Further the quantity σ , connected with h through relations (19) or (20) (obtained by the least squares method), is denoted by σ_{LS} . In Figures 1 and 2 the dependence of $\left[\sigma_{LS}(h)-\sigma_0\right]/\sigma_0$ vs. $2h/\sigma_0$ is plotted by solid lines. As mentioned earlier, in the above calculations σ_0 has been set equal to 1 .

IV. Approximation of $\tilde{g}(x)$ with a Gaussian curve by the y-coordinate of the center of weight

It is easy to check that the one-dimensional Gaussian curve $\tilde{g}^G(x)$ (3) has an ordinate y_G of the center of weight of the area S (S = 1 in our case) given by (Fihtengoltz, 1969)

$$(21) \quad y_G = \frac{S}{4\sqrt{\pi}\,\sigma} \quad ; \qquad (S = 1) .$$

We shall now compute the same quantity y_G , but for the smoothed curve $\tilde{g}(x)$ given by (2), which is not exactly Gaussian one, and by equating this value to y_G from (21) we shall obtain an estimation for σ . The smoothed "observed" curve $\tilde{g}(x)$ is suppoused to be symmetric relatively the axis $x = 0$, and computing y_G we shall restrict us only to the case $x \geq 0$. The y-coordinate of the center of weight of the smoothed curve

$\tilde{g}(x)$ may be expressed as a ratio of two functions of h

(22) $$y_G = A_1(h)/A_2(h) ,$$

where

(23) $$A_1(h) = \int_0^{\tilde{g}(0)} x(\tilde{g})\tilde{g}\, d\tilde{g} ,$$

and

(24) $$A_2(h) = \int_0^{\tilde{g}(0)} x(\tilde{g})\, d\tilde{g} .$$

From (2) we also have

(25) $$d\tilde{g} = \frac{1}{\sqrt{2\pi}\, 2h}\left\{ \exp\left[-\frac{(x+h)^2}{2}\right] - \exp\left[-\frac{(x-h)^2}{2}\right]\right\} dx.$$

As follows from (2),(23) and (25)

(26) $$A_1(h) = (2\pi)^{-1/2}(2h)^{-2}\int_0^{\infty} x\left[\Phi(x+h) - \Phi(x-h)\right]$$
$$\times \left\{ \exp\left[-\frac{(x-h)^2}{2}\right] - \exp\left[-\frac{(x+h)^2}{2}\right]\right\} dx.$$

Let us denote by $L_0(h)$ the expression

(27) $$L_0(h) = \int_{-\infty}^{\infty} \Phi(x+h)\exp(-x^2/2)\, dx .$$

In the following we shall also use the property of the error integral $\Phi(x)$ (Janke et al., 1977)

(28) $$\Phi(x) = 1 - \Phi(-x) .$$

Obviously, according to (26), the quantity $(2\pi)^{1/2}(2h)^2 A_1(h)$ may be expressed as a sum of the four integrals $L_i(h)$ (i = 1,...,4), defined by the next relations

$$(29)\quad L_1(h) = \int_0^\infty x\,\Phi(x+h)\exp\left[-(x-h)^2/2\right]\,dx$$

$$= -\int_0^\infty \Phi(x+h)\,d\exp\left[-(x-h)^2/2\right]$$

$$+ h\int_0^\infty \Phi(x+h)\exp\left[-(x-h)^2/2\right]\,dx\ .$$

Integrating by parts the first term in the right-hand side of the above expression, finally we obtain

$$(30)\quad L_1(h) = \Phi(h)\exp(-h^2/2) + (2\sqrt{2})^{-1}\exp(-h^2/2)$$

$$+ h\int_0^\infty \Phi(x+h)\exp\left[-(x-h)^2/2\right]\,dx\ .$$

Further we have

$$(31)\quad L_2(h) = -\int_0^\infty x\,\Phi(x+h)\exp\left[-(x+h)^2/2\right]\,dx$$

$$= \int_0^\infty \Phi(x+h)\,d\exp\left[-(x+h)^2/2\right]$$

$$+ h\int_0^\infty \Phi(x+h)\exp\left[-(x+h)^2/2\right]\,dx\ .$$

By an analogy with $L_1(h)$, the final result is

$$(32)\quad L_2(h) = -\Phi(h)\exp(-h^2/2) - (2\pi)^{-1/2}\int_0^\infty \exp(x+h)^2\,dx$$

$$+h\int_0^{\infty}\Phi(x+h)\exp\left[-(x+h)^2/2\right]\,dx\,,$$

$$(33)\quad L_3(h)=-\int_0^{\infty}x\,\Phi(x-h)\exp\left[-(x-h)^2/2\right]dx=L_2(-h)\,,$$

and

$$(34)\quad L_4(h)=\int_0^{\infty}x\,\Phi(x-h)\exp\left[-(x+h)^2/2\right]dx=L_1(-h)\,.$$

Consequently

$$(35)\quad \sum_{i=1}^{4}L_i(h)=(2)^{-1/2}\exp(-h^2)+h\left[M_1(h)+M_2(h)\right]$$
$$-(2\pi)^{-1/2}\left\{\int_0^{\infty}\exp\left[-(x+h)^2\right]dx+\int_0^{\infty}\exp\left[-(x-h)^2\right]dx\right\},$$

where the notations

$$(36)\quad M_1(h)=\int_0^{\infty}\Phi(x+h)\exp\left[-\frac{(x-h)^2}{2}\right]dx-\int_0^{\infty}\Phi(x-h)\exp\left[-\frac{(x+h)^2}{2}\right]dx,$$

and

$$(37)\quad M_2(h)=\int_0^{\infty}\Phi(x+h)\exp\left[-\frac{(x+h)^2}{2}\right]dx-\int_0^{\infty}\Phi(x-h)\exp\left[-\frac{(x-h)^2}{2}\right]dx$$

are used. Using the property (28) it may be shown that

$$(38)\quad \int_0^{\infty}\Phi(x-h)\exp\left[-\frac{(x+h)^2}{2}\right]dx=\int_0^{\infty}\left[1-\Phi(h-x)\right]\exp\left[-\frac{(x+h)^2}{2}\right]dx$$
$$=\int_{-\infty}^{-h}\exp(-x^2/2)\,dx-\int_{-\infty}^{0}\Phi(x+h)\exp\left[-(x-h)^2/2\right]dx\,.$$

It is easy to see that

$$(39)\quad M_1(h)=\int_{-\infty}^{\infty}\Phi(x+2h)\exp(-x^2/2)\,dx-\sqrt{2\pi}\,\Phi(-h)$$
$$=L_0(2h)-\sqrt{2\pi}\,\Phi(-h)\,.$$

Analogically, using again the property (28), the second term in the square brackets in the right-hand side of (35) may be computed as follows

$$(40)\quad M_2(h)=\int_{-\infty}^{\infty}\Phi(x)\exp(-x^2/2)\,dx-(2\pi)^{1/2}\left[1-\Phi(-h)\right]$$
$$=L_0(0)-\sqrt{2\pi}+\sqrt{2\pi}\,\Phi(-h)\,.$$

Substituting the results (39) and (40) into (35), and performing the transition from the variable x to $x\sqrt{2}$, and denoting also by $h'=h\sqrt{2}$ in the brace parentheses of the last term of (35), we find that

$$(41)\quad (2\pi)^{1/2}(2h)^2A_1(h)\equiv\sum_{i=1}^{4}L_i(h)$$
$$=(2)^{-1/2}\exp(-h^2)+h\left[L_0(2h)+L_0(0)-\sqrt{2\pi}\right]$$
$$-(4\pi)^{-1/2}\left[\mathcal{N}_1(h')+\mathcal{N}_2(h')\right]\quad,$$

where

$$(42)\quad \mathcal{N}_1(h)=\int_0^{\infty}\exp\left[-(x+h)^2/2\right]dx\;,$$

and

$$(43)\quad \mathcal{N}_2(h)=\int_0^{\infty}\exp\left[-(x-h)^2/2\right]dx=\mathcal{N}_1(-h)\,.$$

There is not difficulties to find analytical expressions for the integrals $\mathcal{N}_1(h')$ and $\mathcal{N}_2(h')$.

$$(44)\quad \mathcal{N}_1(h')=\sqrt{2\pi}\;\Phi(-h')\,.$$

According to (28)

(45) $$\mathcal{N}_1(h') = \sqrt{2\pi} - \sqrt{2\pi}\,\Phi(h') .$$

Therefore

(46) $$(4\pi)^{-1/2}\left[\mathcal{N}_1(h') + \mathcal{N}_2(h')\right] = (2)^{-1/2} .$$

Returning to the definition (27) and differentiating this expression with respect to h ,we have

(47) $$\frac{d\,L_0(h)}{dh} = \frac{\exp(-h^2/2)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-x^2 - hx)\,dx .$$

According to Gradshtein et al.(1963) (formula 3.323.2)

(48) $$\int_{-\infty}^{\infty}\exp(-p_1^2x^2 \pm q_1x)\,dx = \frac{\sqrt{\pi}}{p_1}\exp\left(\frac{q_1^2}{4p_1^2}\right)$$

for $p_1 > 0$.

Substituting in the above expression p_1 = 1 and $q_1 = h$, we obtain the following differential equation for $L_0(h)$

(49) $$\frac{d\,L_0(h)}{dh} = \frac{1}{\sqrt{2}}\exp(-h^2/4) .$$

Its solution is equal to

(50) $$L_0(h) = \sqrt{2\pi}\,\Phi(h/\sqrt{2}) + C ,$$

where C is an arbitrary constant.Using the condition that the error integral is equal to unity at infinity (e.g.,$\Phi(\infty)$= 1), from (50) follows that $L_0(\infty) = \sqrt{2\pi} + C$ .On the other hand from (27) we have $L_0(\infty) = \sqrt{2\pi}$ .Consequently, C = 0.Taking into account that $\Phi(0)$ = 1/2 ,we may write

(51) $$L_o(0) = \sqrt{\pi/2}\ ,$$

(52) $$L_o(2h) = \sqrt{2\pi}\ \Phi(h\sqrt{2})\ ,$$

and

(53) $$L_o(2h) + L_o(0) - \sqrt{2\pi} = \sqrt{2\pi}\ \Phi(h\sqrt{2}) - \sqrt{\pi/2}\ .$$

Substituting (46) and (53) into (41) we obtain the final expression for $A_1(h)$

(54) $$A_1(h) = (2\pi)^{-1/2}(2h)^{-2}\left\{\frac{1}{\sqrt{2}}\exp(-h^2) - \frac{1}{\sqrt{2}} + h\left[\sqrt{2\pi}\ \Phi(h\sqrt{2}) - \sqrt{\pi/2}\right]\right\}.$$

Using (25), the expression (24) for $A_2(h)$ may be rewritten as

(55) $$A_2(h) = -\left(2h\sqrt{2\pi}\right)^{-1}\left[P_1(h) - P_2(h)\right]\ ,$$

where

(56) $$P_1(h) = \int_0^\infty x \exp\left[-(x+h)^2/2\right] dx\ ,$$

and

(57) $$P_2(h) = \int_0^\infty x \exp\left[-(x-h)^2/2\right] dx\ .$$

It is easy to check that

(58) $$P_1(h) = \exp(-h^2/2) - h\,\mathcal{N}_1(h)\ ,$$

and

(59) $$P_2(h) = P_1(-h) = \exp(-h^2/2) + h\,\mathcal{N}_1(-h).$$

Consequently, taking into account (42), (43) and (28)

$$(60)\quad A_2(h) = \left(2h\sqrt{2\pi}\right)^{-1} h\left[\mathcal{N}_1(h)+\mathcal{N}_1(-h)\right] = 1/2.$$

Of course, this result would be expected from our assumption $S = 1$, as mentioned earlier, under the circumstance that the integral (24) gives the half of the area enclosed within $\tilde{g}(x)$ and the abscissa ($y = 0$), and the ordinate ($x = 0$). Equating the right-hand sides of the expressions (21) and (22) we can obtain the value of σ, which we shall denote by σ_{yG}. This means that we have approximated the smoothed quasi-gaussian curve $\tilde{g}(x)$ with an exactly Gaussian curve which has the same y-coordinate of the center of weight as the $\tilde{g}(x)$ do (and also $S = 1$). The final expression for σ_{yG}, according to (55) and (60), is

$$(61)\quad \sigma_{yG} = \frac{h^2}{\sqrt{2}}\left\{\left[\frac{\exp(-h^2)-1}{\sqrt{2}}\right] + h\left[\sqrt{2\pi}\,\Phi(h\sqrt{2}) - \sqrt{\frac{\pi}{2}}\right]\right\}^{-1}.$$

One may check that the passage to the limit $h = 0$ in the above expression gives a reasonable result, e.g., $\lim\limits_{h \to 0} \sigma_{yG}(h) = 1$ (in view of the substitution $\sigma_0 = 1$). This check may be performed by using the L'Hospital's rule for evaluation of indeterminate expressions in terms of the ratio 0/0. In Figures 1 and 2 the dependence of $\left[\sigma_{yG}(h)-\sigma_0\right]/\sigma_0$ vs. $2h/\sigma_0$ is plotted by dashed lines. At first glance, the analytical expressions (19) (or (20)) and (61) giving the dependences between the smoothed values of σ and $2h$ for two different methods of approximation of $\tilde{g}(x)$ are very different. But as can be seen from Figures 1 and 2, the solid and the dashed lines are very close each to other. This result is not surprising because if the width of the scanning slit $2h$ is not too large, the width of the smoothed curve σ is not too different from the width of the unsmoothed curve σ_0. This is not a new conclusion, but the aim of this paper is to get quontitative expressions giving exact analytical connections between   and σ_0 depending on $2h$. In real-time astronomical observations the scanning of turbulent stellar images by a slit "enlarges" their sizes typically by an amount of 0.5 - 1.5 per cent. Such an accuracy of the estimation of σ_0 is useful if a reconstruction of long-exposure turbulent astronomical images is applied.

V. Discussion

The formula (20) is obtained without the assumption that σ is a priori close to σ_0. In the later case this expression may be simplified. Taking into account that $\sigma \gtrsim 1$ ($\sigma_0 = 1$) and using the expansion of the logarithmic function in a series we may write the following approximate expansion (Korn et al., 1977)

$$(62)\qquad \ln\frac{2\sigma^2}{1+\sigma^2} \approx -1 + \frac{2\sigma^2}{1+\sigma^2} = \frac{\sigma^2-1}{\sigma^2+1} \quad ,$$

and substituting this result into (20) to obtain

$$(63)\qquad h = \left[3(1+\sigma_{LS}^2)\ln\frac{2\sigma_{LS}^2}{1+\sigma_{LS}^2}\right]^{1/2} \approx \sqrt{3(\sigma_{LS}^2-1)} \quad .$$

Remembering that in the above formula σ_{LS} and h are measured in units of σ_0 (e.g., it has been assumed that $\sigma_0 = 1$),

(63) may be rewritten as

$$(64) \qquad \sigma_{LS} = \left(\sigma_o^{\,2} + h^2/3\right)^{1/2} ,$$

where σ_o, σ_{LS} and h are evaluated in ordinary units (angular or linear).

Instead of the y_G -coordinate of the area S enclosed by the Gaussian curve $\tilde{g}^G(x)$ it is also possible to use the x-coordinate x_G of the center of weight of this area for $x \geq 0$ (or $|x_G|$ for $x \leq 0$). In place of formula (21) in this case we are able to introduce a new relation which includes the unknown parameter σ (Fihtengoltz, 1969)

$$(65) \qquad x_G = \sqrt{2/\pi}\ \sigma .$$

Note that in the above expression the area S does not appear, e.g., this evaluation is independent of the normalization of $\tilde{g}^G(x)$. Nevertheless, when this coordinate x_G must be computed from the smoothed "observed" curve $\tilde{g}(x)$ it is necessary to decompose the area S into two areas with equal sizes $S_1(x \geq 0) = S_2(x \leq 0) = S/2$. Then we have to estimate the $|x|$ -coordinate of the center of weight for every one part of the area S and to take the arithmetic mean value. So the estimation of x_G will be to some extent dependent from the way by which the area S (enclosed within the smoothed curve $\tilde{g}(x)$ and the axis x) is separated. Such a procedure of decomposition is absent when we are dealing with the determination of σ through the y_G -coordinate. For this reason in this paper we have preferred to evaluate σ (by the method of the center of weight) using only the formula (21) but not the expression (65).

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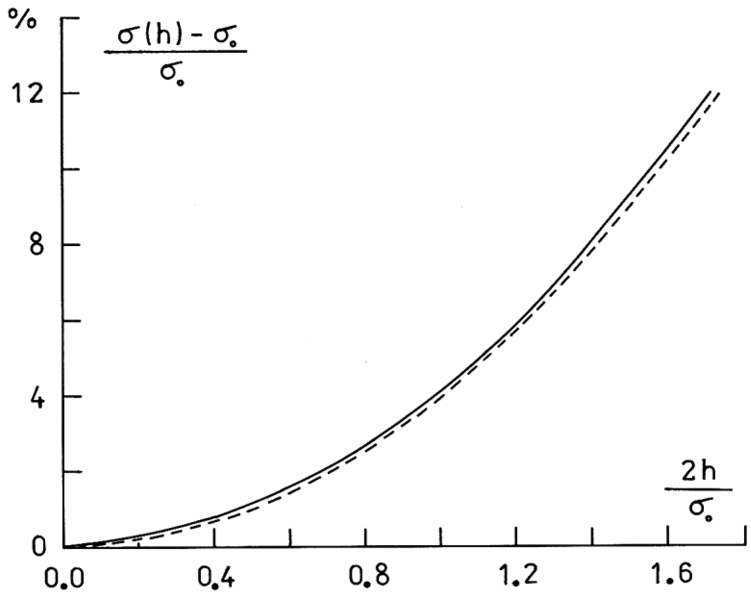

Fig.1. The dependence of $[\sigma_{LS}(h)-\sigma_0]/\sigma_0$ vs. $2h/\sigma_0$ (solid line) and $[\sigma_{yG}(h)-\sigma_0]/\sigma_0$ vs. $2h/\sigma_0$ (dashed line)

Fig.2. Same as Fig.1, but the scale is enlarged