Galaxies, Cosmology and Dark Matter

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Chapter 13

The Homogeneous Universe

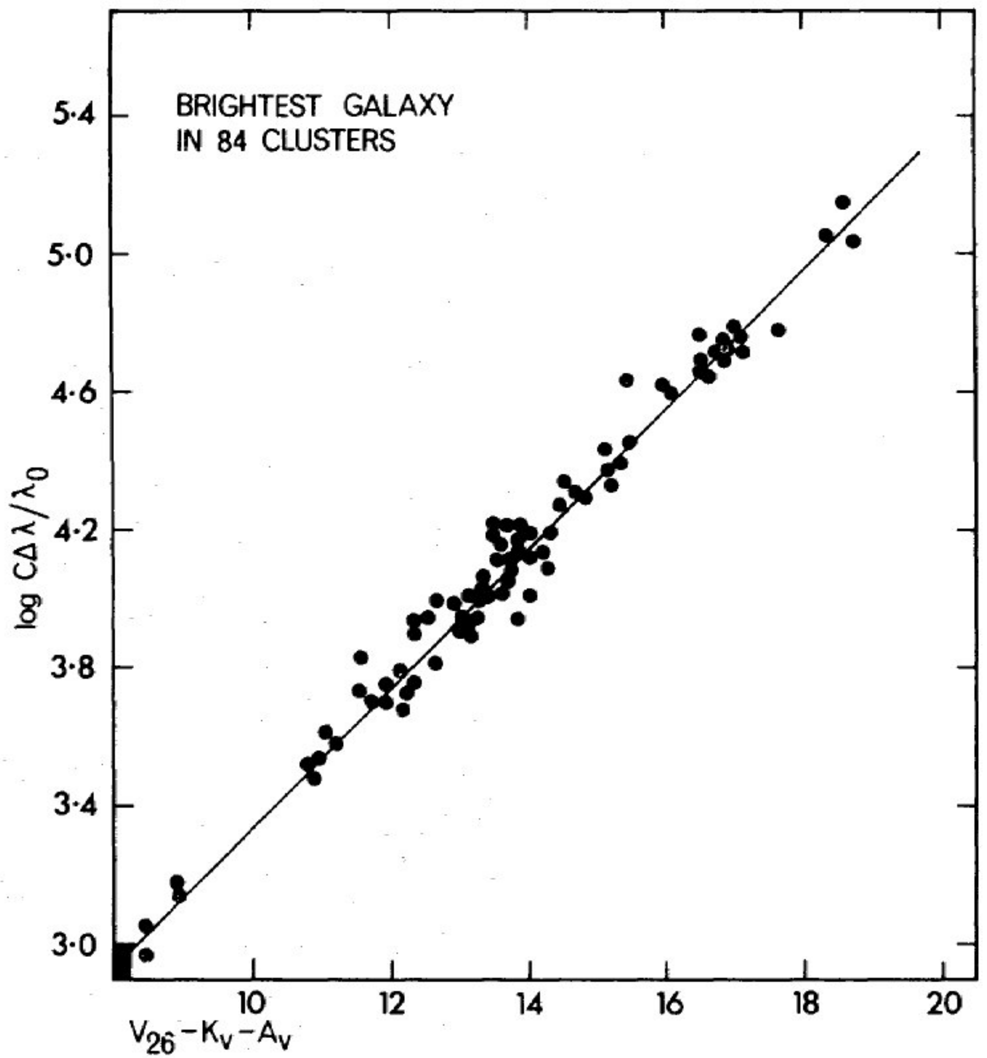
13.1 The Big Bang Scenario: basic observations and assumptions

- Galaxies are observed to move away from each other with their recessional velocity increasing with distance (observation).
- On large scales the universe is close to **isotropic** (distribution of faint galaxies, radio sources, microwave background) though on small scales it evidently is highly anisotropic (observation).
- Our location in the universe and the observations we make are not unique but typical (**cosmological principle** = assumption). Together with the previous item this implies that the universe is **homogeneous** on large scales.
- Gravitation and in turn the dynamics of the universe are governed by the field equations of **Einstein's Theory of General Relativity** (assumption).

The Hubble diagram for first–ranked cluster galaxies:

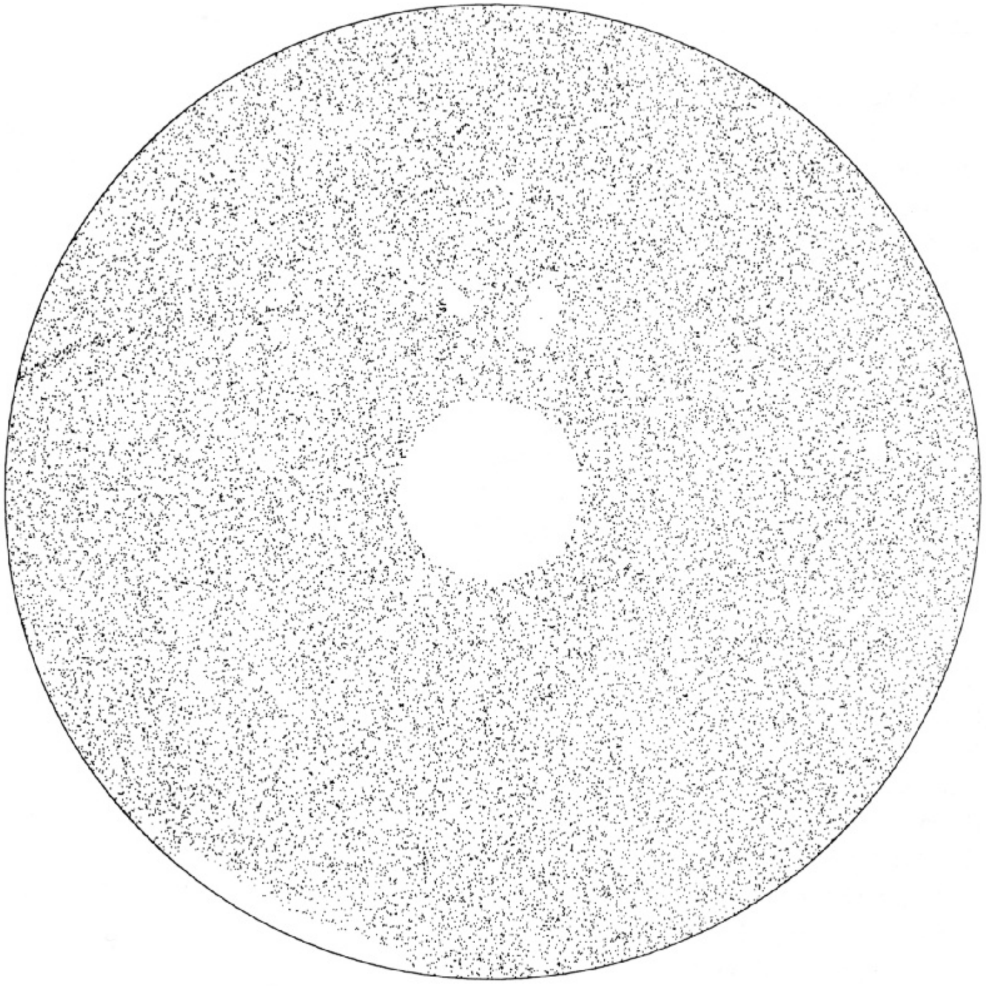
log of redshift vs. apparent V-magnitude

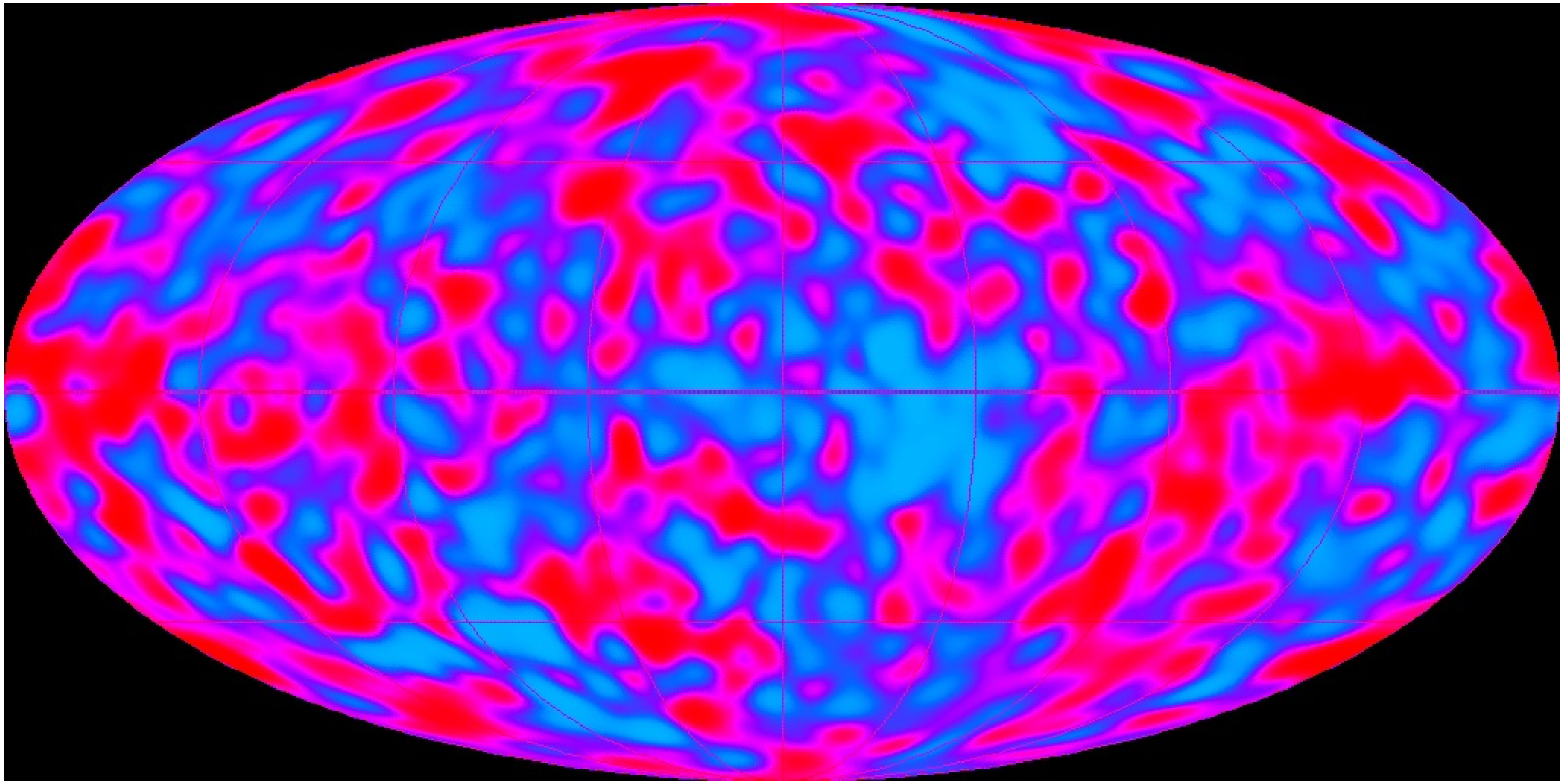
(Sandage (1972) *ApJ*, **178**, 1)



Angular distribution of the $\sim$ 31 000 brightest 6cm radio sources in the sky

(Peebles 1993)





Temperature fluctuations in the Cosmic Microwave Background as measured by the COBE satellite. The amplitude of the fluctuations is only $\Delta T/T \simeq 10^{-5}$ and reflects density inhomogeneities in the baryons of the same order about 100 000 years after the big bang.

General Relativity

The basic equations of **General Relativity** are **Einstein's Field Equations**:

$$\boxed{R_{ij} - \frac{1}{2} g_{ij} \mathcal{R} = 8\pi G T_{ij} + \Lambda g_{ij}} \tag{13.1}$$

R_{ij}:	Ricci tensor ($R_{ij} = R_{ij}(g_{ij})$)	$\leftrightarrow$ space–time curvature
g_{ij}:	metric tensor	$\leftrightarrow$ space–time distances $ds^2 = g_{ij} dx^i dx^j$
$\mathcal{R}$:	Ricci scalar ($\mathcal{R} = g^{ik} R_{ik}$)	$\leftrightarrow$ space–time curvature
G:	gravitational constant	
T_{ij}:	energy–momentum tensor	$\leftrightarrow$ mass, energy, ...
Λ:	cosmological constant	

The Field Equations connect the energy (and thus mass) distribution in space to its geometrical properties (curvature).

For details see e.g. Weinberg, *Gravitation and Cosmology*, J. Wiley 1972, or Misner, Thorne, & Wheeler, *Gravitation*, Freeman 1970.

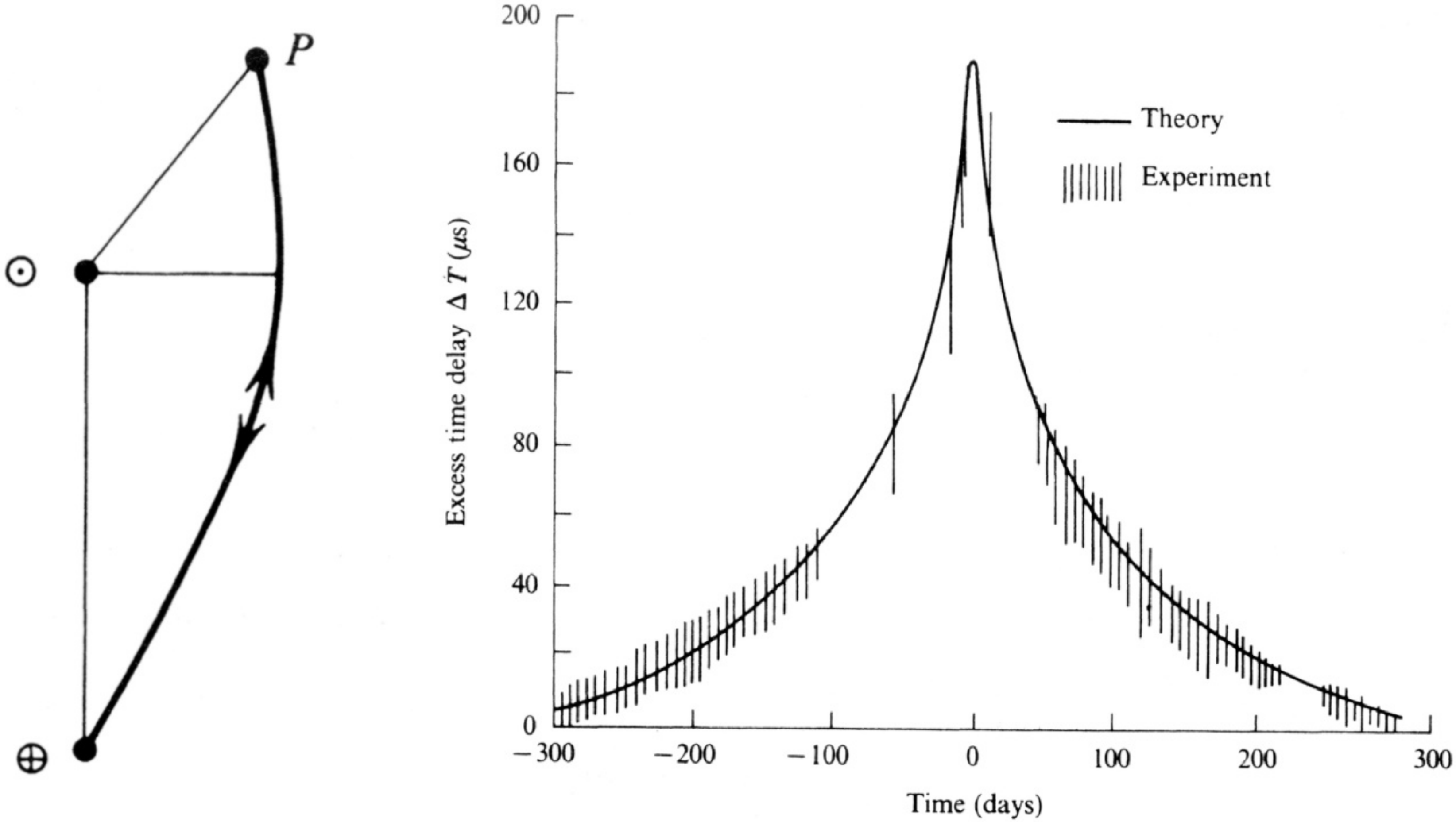
Experimental Tests of General Relativity

- **Perihelion shift.** Orbits of planets around a star (e.g. in the solar system) experience a precession of their perihelia. This was first observed for Mercury and cannot be explained in Newtonian theory. Predictions from General Relativity are in very good agreement with observations:

Planet	a [10^6 km]	e	ϕ [arcsec per century]	
			Observed	Theory
Mercury	57.91	0.2056	43.11 ± 0.45	43.03
Venus	108.21	0.0068	8.4 ± 4.8	8.6
Earth	149.60	0.0167	5.0 ± 1.2	3.8
Icarus	161.00	0.8270	9.8 ± 0.8	10.3

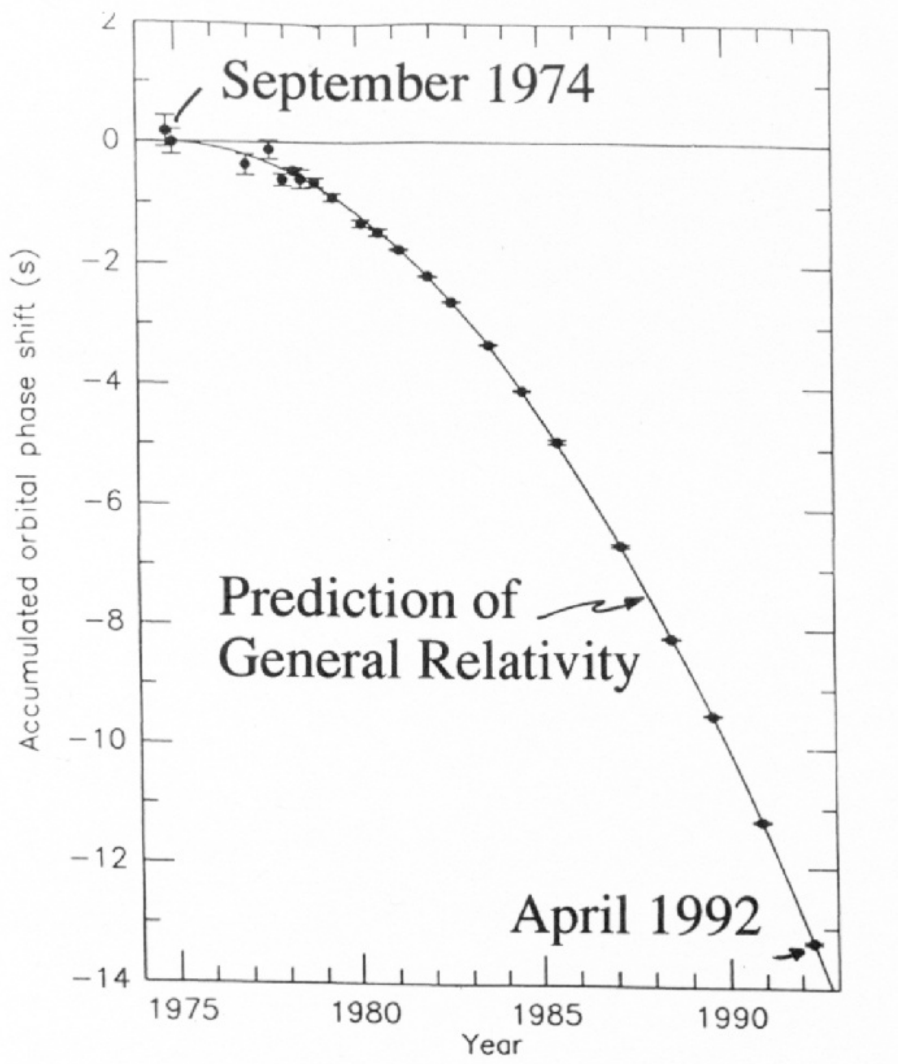
(From Berry, *Principles of Cosmology and Gravitation*, Bristol 1989.)

- **Time delay of signals from Venus.** Radar signals reflected by a planet (e.g. Venus) are delayed when the light path passes the Sun (see figure). Again, the experimental results are in perfect agreement with the theoretical prediction.



(From Berry, *Principles of Cosmology and Gravitation*, Bristol 1989.)

- **Energy loss in binary pulsar.** According to General Relativity, a system of masses orbiting each other loses orbital energy due to the emission of gravitational waves. This loss of orbital energy has been measured accurately for the case of a binary pulsar. (from Longair 1993, *QJRAS*, **34**, 157.)



13.2 The Robertson–Walker Metric

For a homogeneous universe the most general metric for the space–time distance between two points is the **Robertson–Walker metric**:

$$\boxed{ds^2 \;=\; c^2\, dt^2 \;-\; R(t)^2 \left[dr^2 \;+\; R_{c,0}^2 \,\sin^2\left(\frac{r}{R_{c,0}}\right)\,\left(\sin^2\theta\, d\varphi^2 \;+\; d\theta^2\right)\right]} \tag{13.2}$$

with:
- (r, θ, φ): co-moving coordinates
- $R(t)$: cosmic scale factor
- $R_{c,0}$: curvature radius of the universe today

The normalization of r and $R(t)$ can be chosen such that $R(t_0) = 1$ (today!). Then dr describes real distances today, i.e. the co-moving coordinate system preserves present-day distances.

Other versions of the Robertson–Walker Metric

Equation (13.2) is not the only way to write the Robertson–Walker metric. The following form

$$\boxed{ds^2 \;=\; c^2\,dt^2 \;-\; R(t)^2\left[\frac{dr_1^2}{1-k_1r_1^2} \;+\; r_1^2\,\left(\sin^2\theta\,d\varphi^2 \;+\; d\theta^2\right)\right]} \tag{13.3}$$

can be obtained from (13.2) by setting

$$r_1 \;\equiv\; R_{c,0}\,\sin\left(\frac{r}{R_{c,0}}\right) \qquad \text{and} \qquad k_1 \;\equiv\; \frac{1}{R_{c,0}^2}$$

Yet another possibility to write the Robertson–Walker metric is:

$$\boxed{ds^2 \;=\; c^2\,dt^2 \;-\; a(t)^2\left[\frac{dr_2^2}{1-k_2r_2^2} \;+\; r_2^2\,\left(\sin^2\theta\,d\varphi^2 \;+\; d\theta^2\right)\right]} \tag{13.4}$$

which follows by setting

$$a(t) \;\equiv\; R_{c,0}\,R(t) \qquad \text{and} \qquad r_2 \;\equiv\; \frac{r_1}{R_{c,0}} \;=\; \sqrt{k_1}\,r_1 .$$

In this case, $k_2 = -1, 0, +1$ correspond to open, flat, or closed universes, respectively.

13.3 The Friedmann Equations

The geometry of a homogeneous and isotropic universe is described by the g_{ij} of the Robertson–Walker metric (13.2). In order to obtain a solution for the *dynamics* of the universe, the Ricci tensor needs to be calculated from the g_{ij} and the field equations have to be solved for an energy momentum tensor reflecting a homogeneous distribution of mass. For a perfect homogeneous fluid T_{ij} takes the simple form:

$$T_{ij} = \frac{1}{c^2} \begin{pmatrix} \varrho c^2 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

with the density ϱ and the pressure p.

Inserting g_{ij}, R_{ij} and T_{ij} in the field equations (13.1) yields the two **Friedmann equations**:

$$\boxed{\ddot{R} \;=\; -\frac{4\pi G\, R}{3}\,\left(\varrho \,+\, 3\frac{p}{c}\right) \,+\, \frac{1}{3}\,\Lambda\, R} \tag{13.5}$$

$$\boxed{\dot{R}^2 \;=\; \frac{8\pi G\,\varrho}{3}\,R^2 \,+\, \frac{1}{3}\,\Lambda\,R^2 \,-\, \frac{c^2}{R_{c,0}^2}} \tag{13.6}$$

These equations govern the dynamical evolution of the universe (i.e. the time evolution of the scale factor $R(t)$). The Friedmann equations connect this evolution to the intrinsic properties (density ϱ, pressure p, cosmological constant Λ, curvature radius $R_{c,0}$ today) of the universe.

13.4 Basic Cosmological Parameters

The Hubble Constant

The Hubble constant $H(t)$ can be defined in terms of the physical distance $x(t) \equiv R(t)r$ (r: co-moving distance, R: scale factor):

$$\boxed{H \equiv H(t) \equiv \frac{\dot{x}}{x} = \frac{\dot{R}}{R}} \tag{13.7}$$

The value of the Hubble constant today ($t = t_0$) is usually written as $H_0 \equiv H(t_0)$. Note that the Hubble "constant" is **not** constant in time! Since the true value of H_0 is not yet accurately known, it is often parametrized using h:

$$\boxed{H_0 \equiv h\,100\ \text{km s}^{-1}\ \text{Mpc}^{-1}}$$

The current value (based on several methods) is:

$$\boxed{H_0 = 70 \pm 10\ \text{km s}^{-1}\ \text{Mpc}^{-1}}$$

The Density Parameter

The density parameter $\Omega_{m,0}$ is a measure for the matter density ϱ_0 of the universe today and is defined via:

$$\boxed{\Omega_{m,0} \;\equiv\; \frac{\varrho_0}{\varrho_{c,0}} \;\equiv\; \frac{8\pi G \varrho_0}{3H_0^2}} \qquad (13.8)$$

i.e. $\Omega_{m,0}$ is the ratio of the present day matter density to the **critical density** $\varrho_{c,0}$. From various dynamical measurements (galaxies, groups, clusters, large scale galaxy motions) the likely value for $\Omega_{m,0}$ is:

$$\boxed{\Omega_{m,0} \;\simeq\; 0.3}$$

The meaning of Ω_m can be seen after inserting its definition into the Friedmann equation (13.6) and using $R_c = R_{c,0}\, R$:

$$\left(\frac{\dot{R}}{R}\right)^2 \;=\; \frac{8\pi G\varrho}{3} \;-\; \left(\frac{c}{R_c}\right)^2$$

(assuming $\Lambda = 0$), for which we obtain:

$$\big(\Omega_m(t) \;-\; 1\big)\, H(t)^2 \;=\; \left(\frac{c}{R_c}\right)^2$$

Thus the different values of $\Omega_{m,0}$ imply:

$\Omega_{m,0} < 1$	$R_c^2 < 0$	negative curvature	open universe
$\Omega_{m,0} > 1$	$R_c^2 > 0$	positive curvature	closed universe
$\Omega_{m,0} \rightarrow 1$	$R_c^2 \rightarrow \infty$	no curvature	flat (Euclidean) universe

The Deceleration Parameter

The deceleration parameter is defined in the following way:

$$\boxed{q_0 \equiv - \left(\frac{\ddot{R}\, R}{\dot{R}^2} \right)_0} \tag{13.9}$$

Relations between the Cosmological Parameters

The Friedmann equation (13.5) reads for zero pressure ($p = 0$):

$$\ddot{R} = -\frac{4\pi G\,R}{3}\,\varrho + \frac{1}{3}\Lambda\,R.$$

Multiplying by $R^2/\dot{R}^2$ yields

$$\frac{\ddot{R}R}{\dot{R}^2} = -\Omega_m\,\frac{H^2}{2}\,\frac{R}{\dot{R}^2} + \frac{1}{3}\Lambda\,\frac{R^2}{\dot{R}^2}$$

Using the definition of q and Ω_m and defining $\Omega_\Lambda = \frac{1}{3}\Lambda/H^2$ results in:

$$\boxed{q = \frac{\Omega_m}{2} - \Omega_\Lambda \qquad (p = 0)}$$

Dividing the second Friedmann equation (13.6) by R^2 leads to:

$$\boxed{\frac{c^2}{H^2 R_c^2} = \Omega_m + \Omega_\Lambda - 1}$$

where we have used the definition of the Hubble constant.

13.5 The Redshift

The cosmological redshift z is defined as:

$$\boxed{z \equiv \frac{\lambda_o - \lambda_e}{\lambda_e}} \tag{13.10}$$

where λ_e is the rest-frame wavelength of the emitted radiation, and λ_o is the observed wavelength.

The redshift z can be related to the scale factor $R(t)$ by the following considerations.

Assume that a source at co-moving coordinates $(r_e, \varphi_e, \theta_e) =$ const emits two signals (e.g. two maxima of an electromagnetic wave) at times t_e and $t_e + \tau_e$, and that these signals are observed by an observer at $(r_o, \varphi_o, \theta_o =$ const) at times t_o and $t_o + \tau_o$.

The propagation of light is described by $ds^2 = 0$, i.e. the Robertson–Walker metric reads:

$$\frac{c\,dt}{R(t)} = dr$$

where we have chosen our coordinate system such that $r_o = 0$ and $d\Omega = 0$.

Integrating along the trajectories of both signals yields:

$$\int_{t_e}^{t_o} \frac{c\,dt}{R(t)} = \int_{r_e}^{0} dr = \int_{t_e+\tau_e}^{t_o+\tau_o} \frac{c\,dt}{R(t)}.$$

The integrals in time can be re-written in the following way:

$$\int_{t_e}^{t_e+\tau_e} \frac{c\,dt}{R(t)} + \int_{t_e+\tau_e}^{t_o} \frac{c\,dt}{R(t)} = \int_{t_e+\tau_e}^{t_o} \frac{c\,dt}{R(t)} + \int_{t_o}^{t_o+\tau_o} \frac{c\,dt}{R(t)},$$

where two of the integrals cancel. For short time intervals τ_e and τ_o we can assume $R(t) \simeq$ const. Thus:

$$\frac{\tau_e}{R(t_e)} = \frac{\tau_o}{R(t_o)},$$

If we choose τ to be the period of a light wave: $\lambda = c\tau$, we finally obtain:

$$\boxed{\frac{R(t_o)}{R(t_e)} = \frac{\lambda_o}{\lambda_e} = 1 + z}$$

13.6 Dynamics of Homogeneous Universes

Universes with $p = 0$ and $\Lambda = 0$

In this case, the Friedmann equations read:

$$\ddot{R} = -\frac{4\pi G R}{3}\,\varrho$$
$$\dot{R}^2 = \frac{8\pi G R^2}{3}\,\varrho - \frac{c^2}{R_{c,0}^2}$$

From the conservation of mass we have

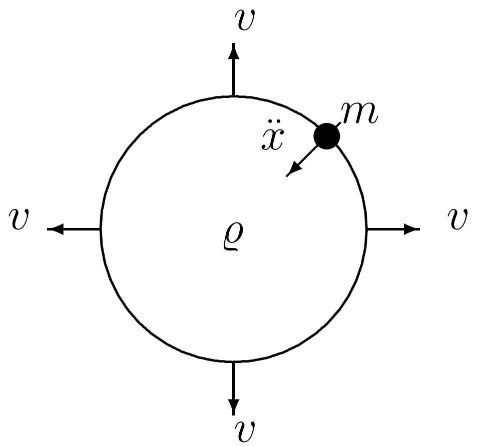
$$\varrho = \varrho(t) = \varrho_0 \left(\frac{R_0}{R}\right)^3 = \varrho_0\,\frac{1}{R^3}$$

(remember that $R_0 = 1$!), and thus:

$$\ddot{R} = -\frac{4\pi G \varrho_0}{3}\,\frac{1}{R^2}$$
$$\dot{R}^2 = \frac{8\pi G \varrho_0}{3}\,\frac{1}{R} - \frac{c}{R_{c,0}^2}$$

(*Note:* the second equation is just the integral of the first one. This also means that the conservation of mass follows from the two Friedmann equations).

Interestingly, these relations are identical to those derived in classical mechanics for a homogeneous, expanding sphere:



In this case, the equation of motion has the form:

$$m\,\ddot{x} = -\frac{GMm}{x^2} = -\frac{4\pi G\varrho}{3}x$$

Using a co-moving coordinate system: $\varrho = \varrho_0 R^{-3}$, $x = x_0 R$, we can rewrite this equation as:

$$\ddot{R} = -\frac{4\pi G \varrho_0}{3} \frac{1}{R^2}$$

which is identical to the Friedmann equation. This result which has been derived *locally* with Newtonian dynamics turns out to be the same as the one which is valid for the universe as a whole. The reason lies in the homogeneity of the Universe. As each part will expand in the same way, it is sufficient to consider the dynamics of a small volume only to derive the equation of motion. For a small volume, however, space becomes flat and the finite velocity of light can be neglected, which means that a Newtonian approach gives the same equations of motion as the Friedmann model.

Einstein–de-Sitter Universe

This is a universe with $\Omega_m = 1$, $\Omega_\Lambda = 0$, i.e. the universe is Euclidean:

$$\dot{R}^2 = \frac{8\pi G\varrho}{3} R^2$$

which can be integrated and yields:

$$R^{1/2}\, dR = \left(\frac{8\pi G\varrho_0}{3}\right)^{1/2} dt$$

Using the definition of Ω_m (13.8) and considering that we assumed $\Omega_m = 1$, we have $H_0^2 = (8\pi G\varrho_0)/3$ and thus:

$$\boxed{R = \left(\frac{3}{2}\, H_0\, t\right)^{2/3} \qquad (p = 0, \Lambda = 0, \Omega_m = 1)}$$

From $R_0 = 1$ today, we can solve this equation for the **age of universe** t_0:

$$\boxed{t_0 = \frac{2}{3}\frac{1}{H_0} \qquad (p = 0, \Lambda = 0, \Omega_m = 1)}$$

Thus, for $p = 0$, $\Lambda = 0$, and $\Omega_m = 1$, we have

$$H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1} \rightsquigarrow t_0 \simeq 13 \text{ Gyr}$$
$$H_0 = 100 \text{ km s}^{-1} \text{ Mpc}^{-1} \rightsquigarrow t_0 \simeq 7 \text{ Gyr}$$

Empty Universe

For $\Omega_m \to 0$ and $\Lambda = 0$ we have $\varrho_0 \to 0$, and the Friedmann equation (13.6) becomes:

$$\frac{c^2}{R_c^2} = -H^2 \quad \Leftrightarrow \quad \dot{R}^2 = H_0^2$$

and thus:

$$\boxed{R = H_0 \, t \quad \text{and} \quad t_0 = \frac{1}{H_0}}$$

from which we obtain for the age of the universe:

$$H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1} \rightsquigarrow t_0 \simeq 20 \text{ Gyr}$$
$$H_0 = 100 \text{ km s}^{-1} \text{ Mpc}^{-1} \rightsquigarrow t_0 \simeq 10 \text{ Gyr}$$

Since the cosmological parameters can be restricted to the ranges $0.1 \leq \Omega_m \leq 1$ and 50 km s^{-1} Mpc^{-1} $\leq H_0 \leq 100$ km s^{-1} Mpc^{-1} we can constrain:

$$\boxed{7 \text{ Gyr} \leq \text{ age of the universe } t_0 \leq 20 \text{ Gyr}}$$

Universes Dominated by the Cosmological Constant

In this case, we can neglect the matter density: $\varrho \simeq 0$. Then the second Friedmann equation (13.6) reads

$$\dot{R}^2 = \frac{1}{3}\Lambda R^2 - \frac{c^2}{R_{c,0}^2}. \qquad (13.11)$$

This equation has the solution:

$$R = \frac{c}{|R_{c,0}|\sqrt{\Lambda/3}} \sinh\left(\sqrt{\frac{\Lambda}{3}}\, t\right).$$

Since $\sinh x = 1/2(e^x - e^{-x})$ this is equivalent to exponential expansion of the universe:

$$R \propto \exp\left(\sqrt{\frac{\Lambda}{3}} t\right) \quad (\Lambda \text{ dominated universe})$$

In this case, we will have $\Lambda R^2/3 \gg c^2/R_{c,0}^2$ after a short period of time, and thus equation (13.11) reduces to

$$\dot{R}^2 = \frac{1}{3}\Lambda R^2.$$

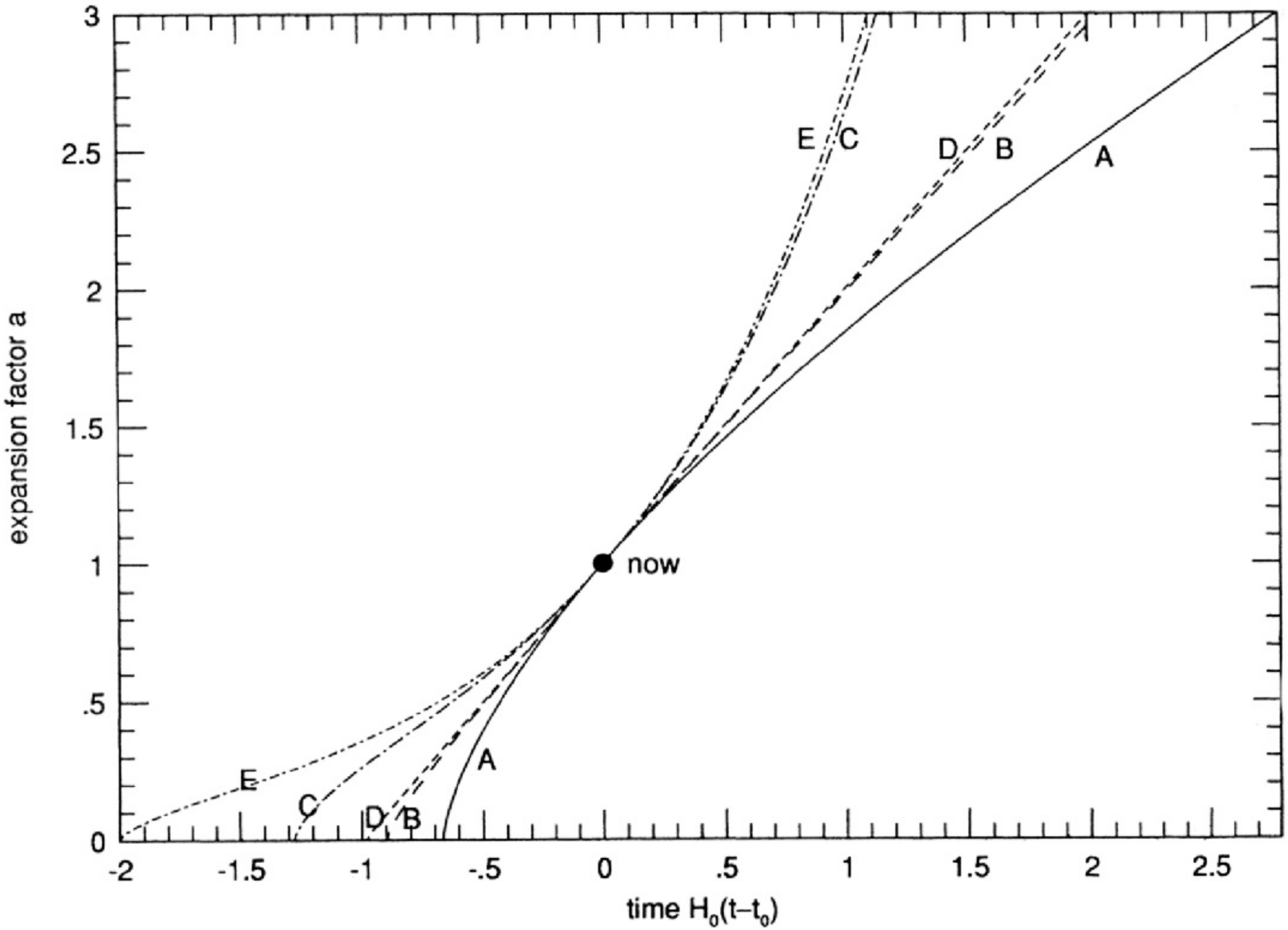
This means that the curvature of space becomes negligible, and space–time becomes Euclidean.

In the following figure, taken from Carroll, Press, & Turner, 1992, *ARAA*, **30**, 499, we show the dynamical history of five cosmological models with the following parameters:

Model	Ω_{tot}	Ω_m	Ω_Λ	Description
A	1	1	0	flat, matter dominated, no Λ
B	0.1	0.1	0	open, plausible matter, no Λ
C	1	0.1	0.9	flat, Λ plus plausible matter
D	0.01	0.01	0	open, minimal matter, no Λ
E	1	0.01	0.99	flat, Λ plus minimal matter

where the symbols are defined as follows:

$$\Omega_{tot} \equiv \Omega_m + \Omega_\Lambda \quad \text{with} \quad \Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2}$$


3
2.5
2
1.5
1
.5
0
expansion factor a
-2
-1.5
-1
-.5
0
.5
1
1.5
2
2.5
time $H_0(t-t_0)$
now
A
B
C
D
E

13.7 Evolution of Physical Properties

13.7.1 Intrinsic Luminosity and Observed Flux

To relate the intrinsic luminosity of an object (e.g. a galaxy) at redshift z to its observed flux, we use the fact that $ds^2 = 0$ for the propagation of light. Then the Robertson–Walker metric (13.4) reads

$$0 = ds^2 = c^2\,dt^2 - a(t)^2\,\frac{dr^2}{1-kr^2},$$

where we have set $k_2 = k$, $r_2 = r$ (co-moving), and $d\Omega = 0$. This equation can be integrated to give

$$\int_{t_1}^{t_0} \frac{c\,dt}{a(t)} = \int_{r_1}^{0} \frac{dr}{(1-kr^2)^{1/2}} \tag{13.12}$$

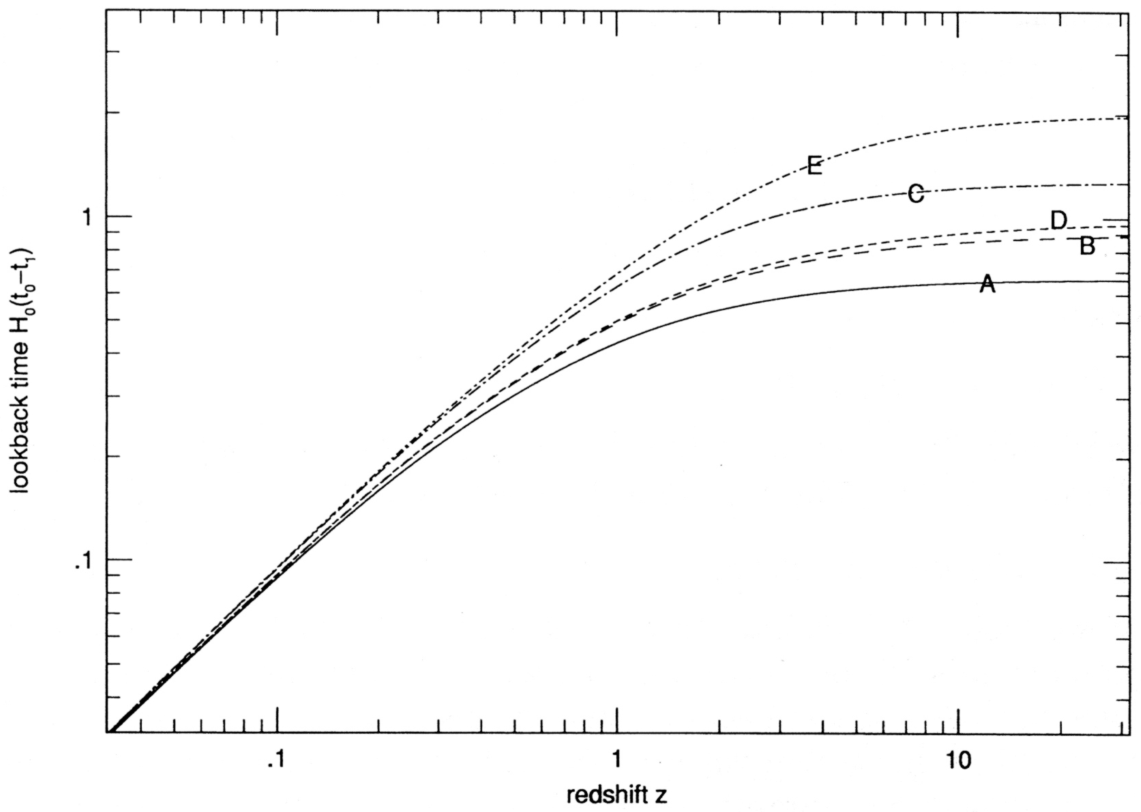
where we have assumed that the object emits its radiation at r_1 and time t_1. For small distances, $a(t)$ can be expanded in a series:

$$a(t) = a(t_0)\left\{1 + \left(\frac{\dot{a}}{a}\right)_{t_0}(t-t_0) + \frac{1}{2}\left(\frac{\ddot{a}}{a}\right)_{t_0}(t-t_0)^2 + \ldots\right\}$$
$$= a(t_0)\left\{1 + H_0(t-t_0) + \frac{1}{2}q_0 H_0^2 (t-t_0)^2 + \ldots\right\}$$

Using $1+z = a(t_0)/a(t)$ we can write this as

$$t - t_0 = \frac{1}{H_0}\left\{z - (1 + \frac{q_0}{2})\, z^2 + \ldots\right\}$$

The expression $t - t_0$ is called the **look–back time**. Its redshift evolution for the five cosmological models defined on page 484 is shown in the following figure:



(From Carroll, Press, & Turner (1992) *ARAA*, **30**, 499)

Then integration of (13.12) yields:

$$x_1 = a_0 \, r_1 = \frac{c}{H_0} \left\{ z - \frac{1}{2}(1 + q_0) z^2 + \ldots \right\}$$

This, of course, is the present-day distance of the galaxy which emitted the light at a redshift z , i.e. when the Universe was a factor $(1+z)$ smaller.

To derive an expression for the observed flux of the galaxy, we have to consider three effects:

1. Time intervals δt_1 (e.g. between subsequent maxima of an electromagnetic wave) arrive at the observer in an interval $\delta t_o = a_0/a(t_1)\delta t_1$ due to the redshift (time–dilation) effect.
2. The spatial distance between subsequent photons is increased by a factor of $(1+z)$.
3. The photons are distributed on a surface of area $4\pi x_1^2 = 4\pi a_0^2 r_1^2$.

Thus the observed flux ℓ from a source with intrinsic luminosity L at a redshift z is

$$\begin{aligned}\ell &= \frac{L}{4\pi a_0^2 r_1^2 (1+z)^2} \\ &= \frac{L}{4\pi \left\{ cH_0^{-1} \left[z + \frac{1}{2}(1-q_0)z^2 + \ldots \right] \right\}^2}\end{aligned}$$

If we now define the **luminosity distance** d_L such that the simple relation

$$\boxed{\ell \equiv \frac{L}{4\pi d_L^2}} \quad (13.13)$$

is valid, we can write d_L as

$$\boxed{d_L = \frac{c}{H_0} \left[z + \frac{1}{2}(1-q_0)z^2 + \ldots \right]} \quad (13.14)$$

Mattig (1958!) derived a closed analytic expression for the luminosity distance d_L (in the case of $\Lambda = 0$):

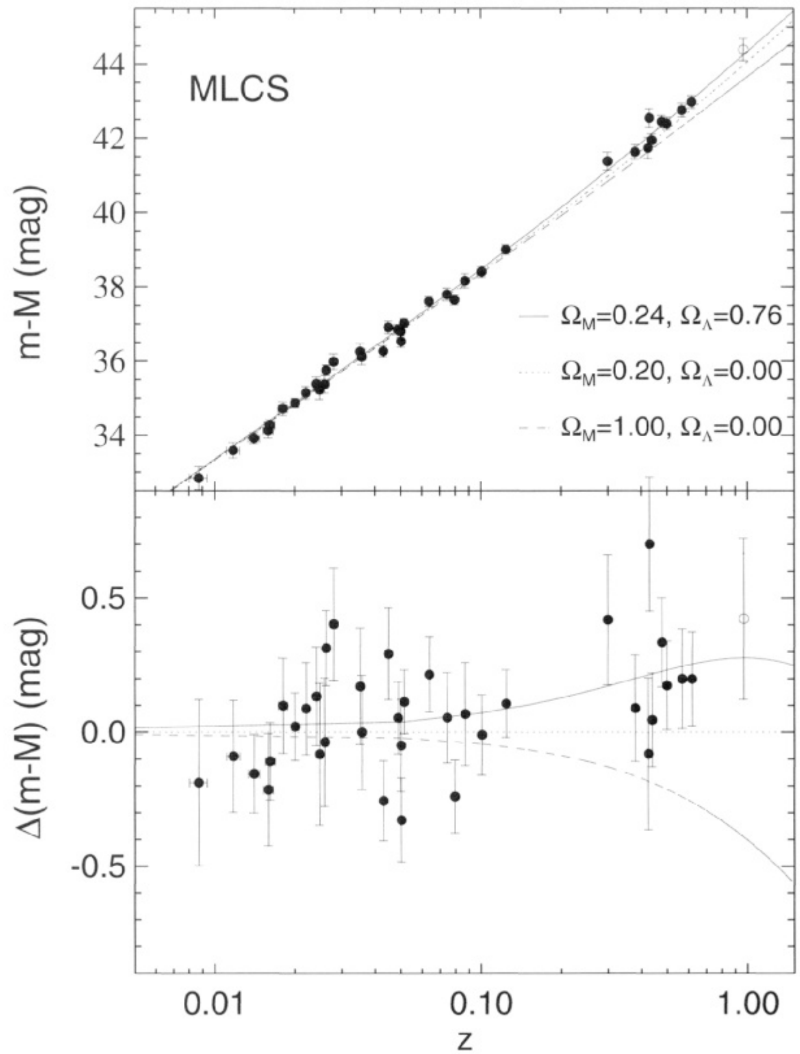
$$\boxed{\begin{aligned} d_L = & \ \frac{c}{H_0} \frac{q_0 z + (q_0 - 1)\left(\sqrt{1 + 2q_0 z} - 1\right)}{q_0^2} \\ = & \ \frac{cz}{H_0} \left\{ \frac{1 + z + \sqrt{1 + 2q_0 z}}{1 + q_0 z + \sqrt{1 + 2q_0 z}} \right\} \end{aligned}} \tag{13.15}$$

Measuring q_0 using d_L:

The first measurements of q_0 were based on cluster galaxies assumed to be standard candles, e.g. the first–ranked cluster galaxies (Sandage 1972). The problem with this approach is that galaxies evolve with redshift and that it is extremely difficult to disentangle evolution effects from different world models. Modern determinations of q_0 use Type Ia Supernovae as standard candles. Very likely, these are true standard candles (but this has not been demonstrated rigorously either). The following figures, taken from Riess *et al.*, 1998, *AJ*, **116**, 1009, show the Supernova Ia Hubble diagram and the confidence ranges for the cosmological parameters. There are strong indications that the Universe is flat and that the cosmological constant may be non-zero.

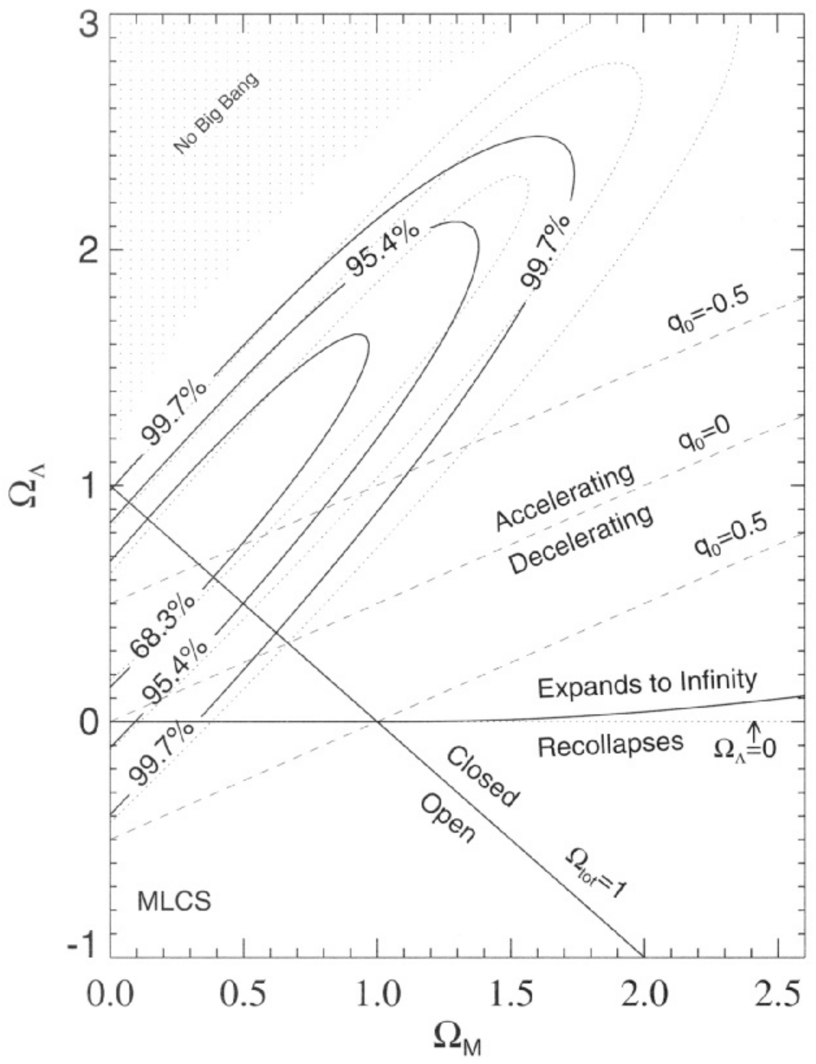
Supernova Ia Hubble diagram

(Riess *et al.* 1998)



Supernova Ia: Resulting parameters

(Riess *et al.* 1998)



13.7.2 Physical Sizes and Angular Diameters

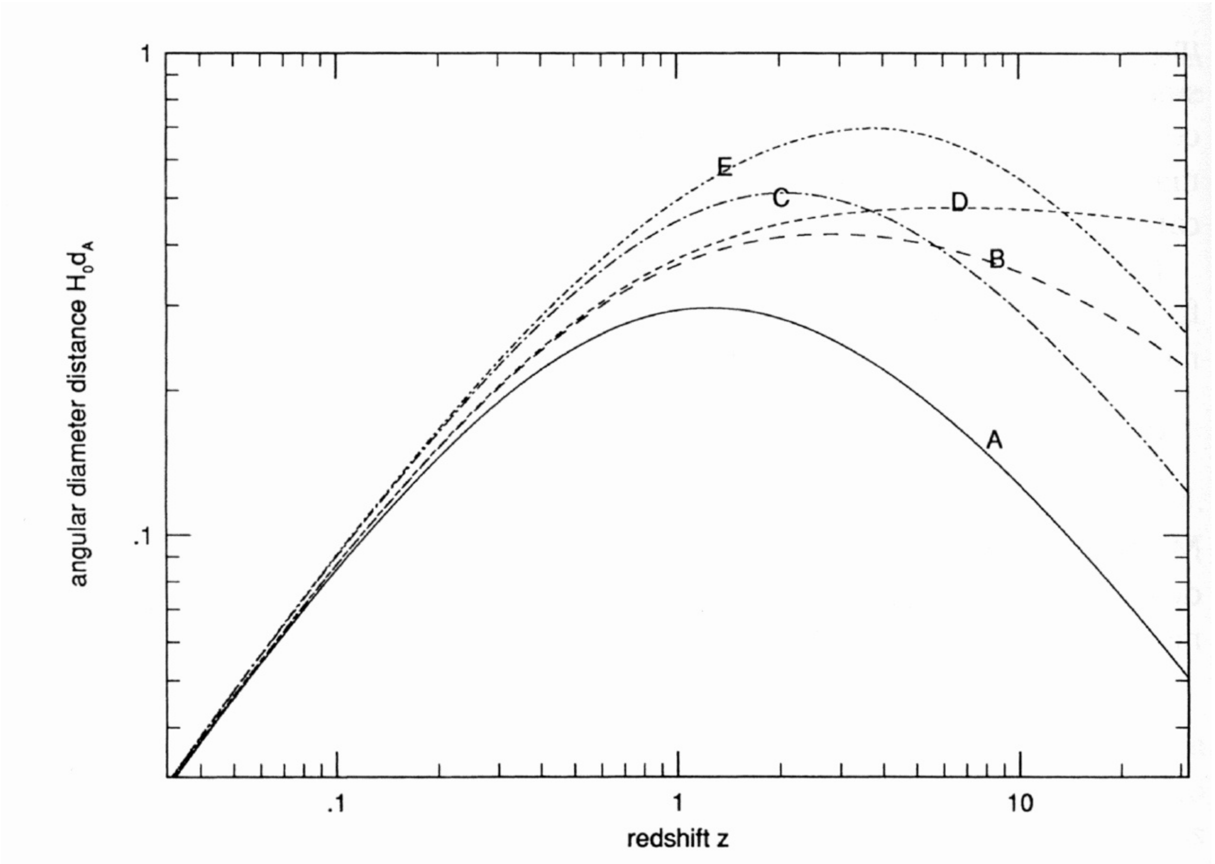
The angular diameter δ of a source with physical diameter D at redshift z is determined by the distance between source and observer at the time t_1 of light emission. Thus:

$$\delta = \frac{D}{x_1} = \frac{D}{a(t_1)r_1}.$$

Using $a(t_1) = a_0/(1+z)$ we can define the angular diameter δ in terms of the **angular diameter distance** d_A:

$$\boxed{\begin{aligned} \delta &\equiv \frac{D}{d_A} \\ d_A &= \frac{d_L}{(1+z)^2} \end{aligned}} \qquad (13.16)$$

with the luminosity distance d_L defined from (13.15).

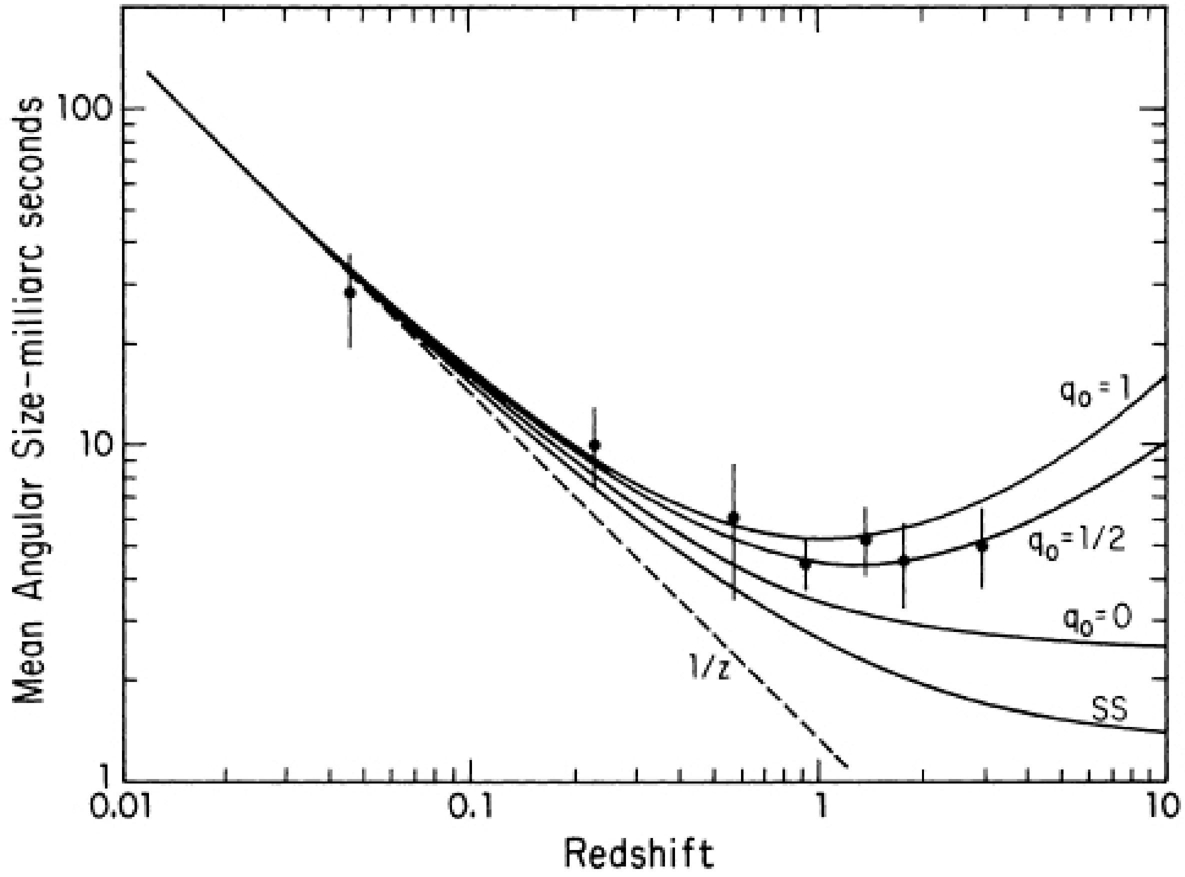


(From Carroll, Press, & Turner, 1992, *ARAA*, **30**, 499
models defined on page 484)

Measuring q_0 using d_A:

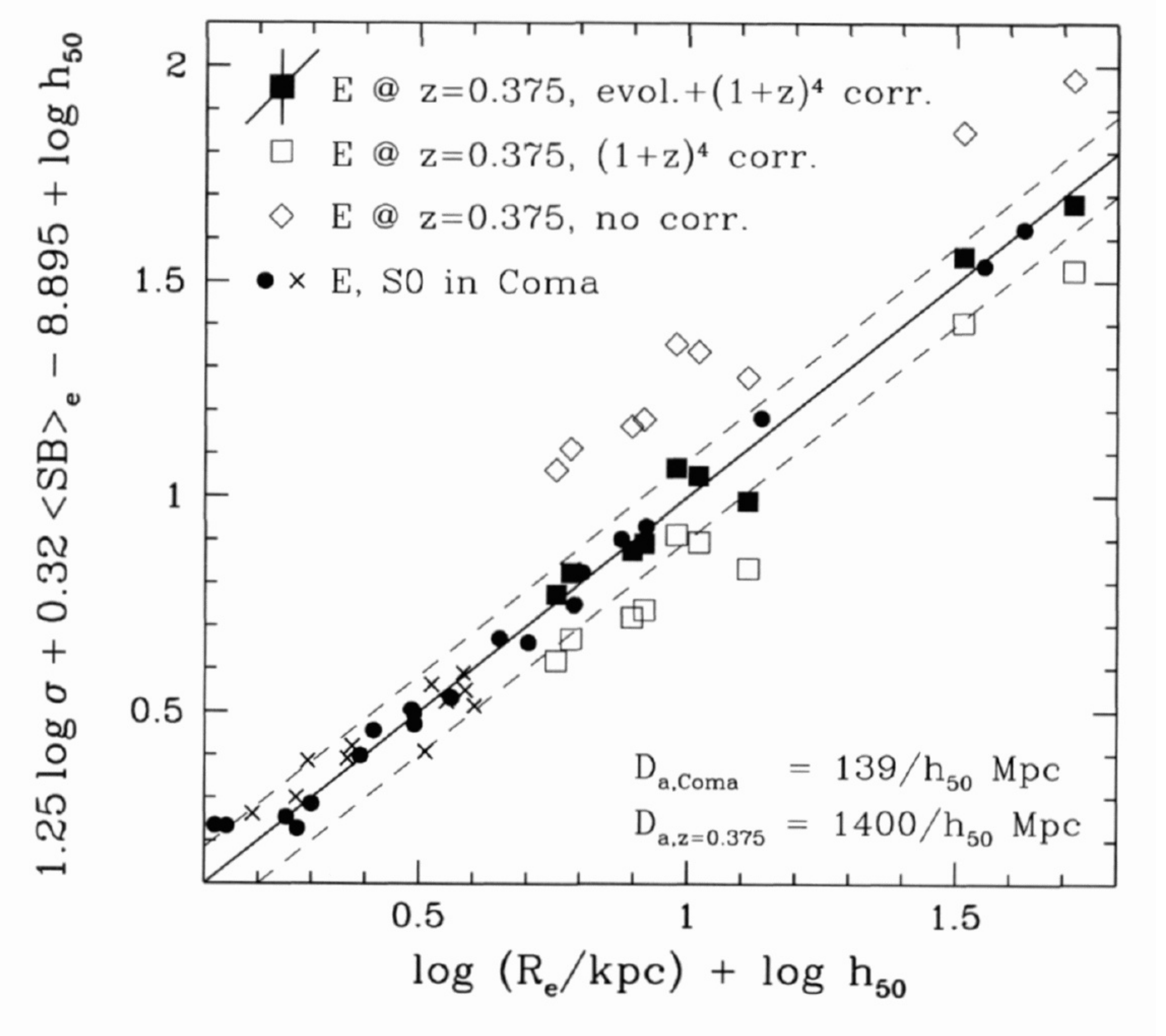
To use these relations to determine q_0, one needs so-called standard rods. These must have the same physical diameter at all redshifts, or, it must be possible to calculate their physical diameter without knowing their distance.

In the past, several classes of objects have been used as standard rods, e.g. compact radio sources. The following figure shows an example. (*Note:* there exist many reasons why the physical diameter of compact radio sources is probably not constant with redshift!!)

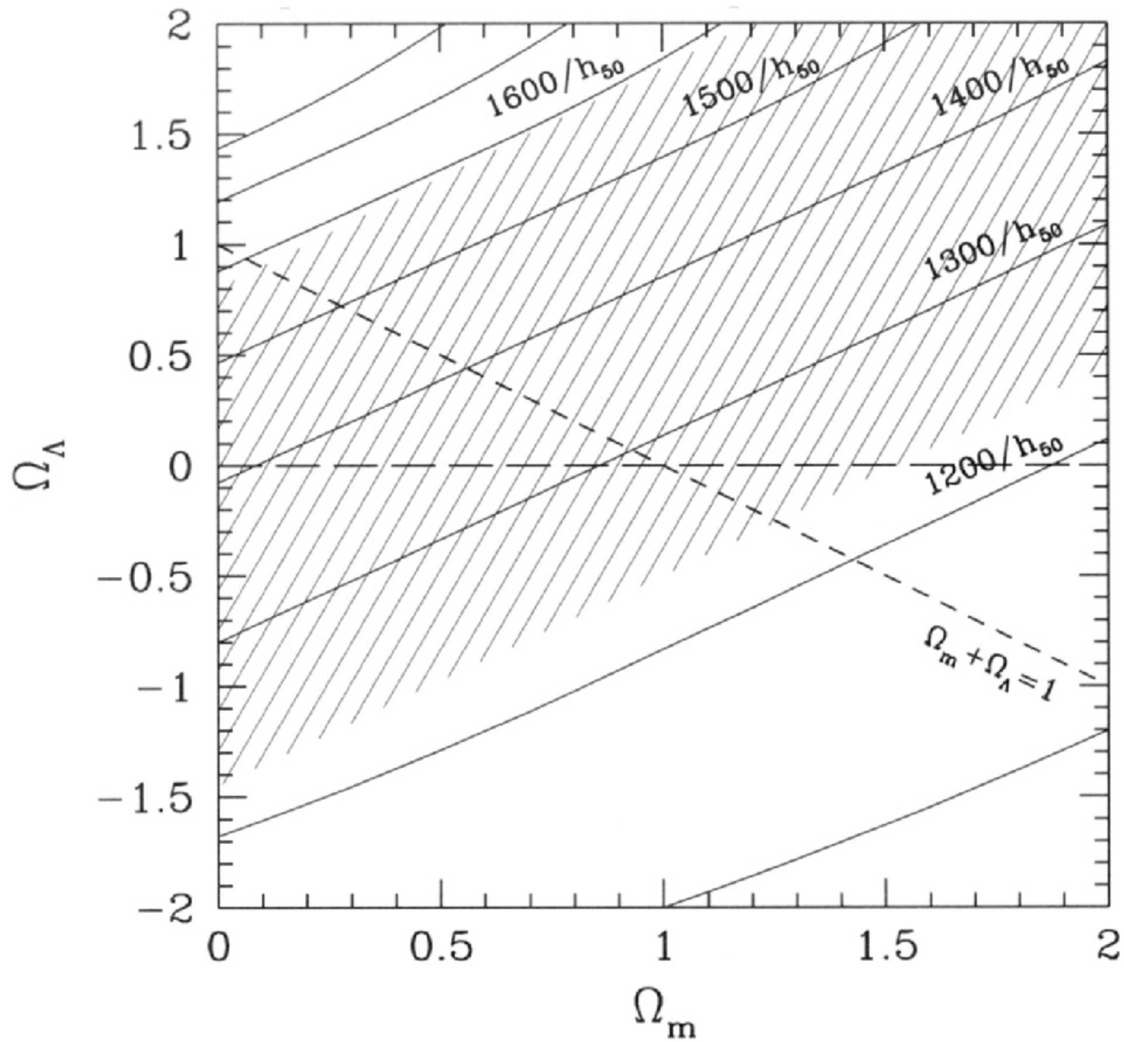


(Kellermann 1993. In: Observational Cosmology. ASP Conference Series, **51**, 50)

A comparison of physical effective radii of elliptical galaxies in the $z = 0.375$ galaxy cluster Abell 370 and the local Coma cluster. The required distance to match the distant objects to the local ones constrains the geometry of the universe.
Bender *et al.* 1998,
ApJ, **493**, 529



Bender *et al.* 1998,
ApJ, **493**, 529



13.7.3 Surface Brightness

In a Newtonian, non-expanding universe, surface brightnesses are independent of distance. The surface brightness Σ in an expanding Friedmann model is given by:

$$\Sigma = \frac{\ell}{\pi\delta^2} = \frac{\text{observed flux}}{\text{square arcsec}}$$

for a circular source of constant surface brightness. Inserting the definitions for the luminosity distance d_L and the angular diameter distance d_A yields:

$$\Sigma = \frac{L/4\pi d_L^2}{\frac{\pi D^2}{d_L^2}(1+z)^4},$$

satisfying $\Sigma(z=0) = L/(4\pi^2 D^2)$.

Thus the surface brightness of an object at redshift z obeys the so-called **Tolman relation:**

$$\boxed{\Sigma(z) = \frac{\Sigma(z=0)}{(1+z)^4}} \qquad (13.17)$$

Note that the surface brightness does not depend on q_0 and thus on the dynamical evolution of the universe!

$\mu_K =$ surface brightness in the K-band (2.2μm)
Pahre *et al.*, 1996,
ApJ, **456**, L79