Galaxies, Cosmology and Dark Matter

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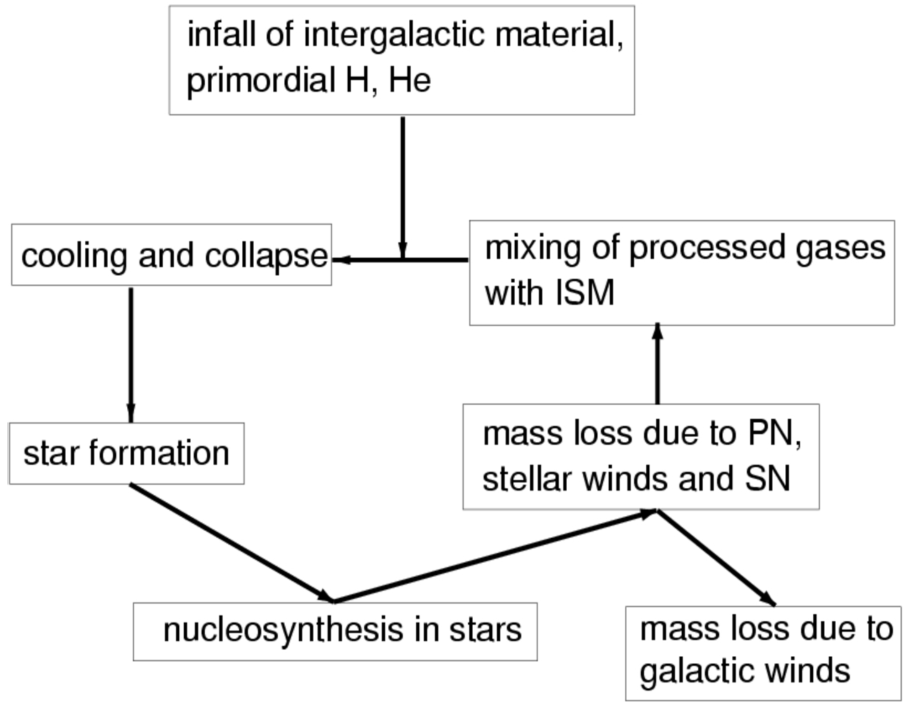
Chapter 3

Chemical Evolution

Key parameters in chemical evolution:

- Lifetimes of stars (as a function of mass)
- Mass distribution of stars at their birth
- Star formation rate
- Element production of stars
- Ejection mechanisms
- Mixing with interstellar gas
- Interaction with environment (gas inflow/outflow)



more elaborate descriptions in, e.g.:
Pagel: Nucleosynthesis and Chemical Evolution of Galaxies, Cambridge Univ. Press
Tinsley: Fundamentals of Cosmic Physics, Vol. 5, 1980

Table 7.1. *Representative stellar data*

$M_{\text{init.}}$ ($M_\odot$)	On zero-age main sequence (ZAMS): log $L/L_\odot$	T_{eff} (10^3K)	Sp	M_V	$B-V$	τ_{ms} Myr or Gyr[a]	τ_{tot} Myr or Gyr[b]
120	6.2	53(59)				3.0(2.9)	3.9(3.2)
60	5.7	48(52)				3.5(3.9)	4.0(4.3)
40	5.4	44(48)	O5	−5.6	−0.32	4.4(5.1)	5.0(5.5)
20	4.6	35(39)	O8	−5.0	−0.31	8.2(9.4)	9.0(10.2)
12	4.0	28(32)	B0.5	−4.0	−0.28	16(18)	18(20)
7	3.3	21(25)	B2	−1.5	−0.22	43(45)	48(50)
5	2.8	17(21)	B4	−0.8	−0.19	94(88)	107(100)
3	1.9(2.1)	12.2(16.1)	B7	−0.2	−0.12	350(290)	440(340)
2	1.2(1.4)	9.1(12.2)	A2	1.4	0.05	1.16(.86)	1.36(1.03)
1.5	.68(.92)	7.1(9.6)	F3	3.0	0.40	2.7(1.84)	(2.0)
1.0	−.16(.15)	5.64(6.71)	G5	5.2	0.65	10.0(7.3)	
0.9	−.39(−.07)	5.30(6.31)	K0	5.9	0.89	15.5(10.7)	
0.8	−.61(−.31)	4.86(5.86)	K2	6.4	0.94	25(15)	

[a] Time to end of core H burning.
[b] Time to end of C or He burning.
Sources: Schaller *et al.* (1992); Meynet *et al.* (1994); Tinsley (1980); Allen (1973).

see: Pagel (1993) *Instituto Astrofisica di Canarias Winter School*

3.1 Production and Emission of the Elements

Stars with $M \geq 10 M_\odot$ dominate element production.

Stages of nuclear burning for a star with **25** $M_\odot$:

Stages of nuclear burning	Temperature ($[10^9\,K]$)	Duration	Main Products
hydrogen burning	0.06	$7 \cdot 10^6\,yr$	He
helium burning	0.23	$5 \cdot 10^5\,yr$	C (O, Ne, Mg, Si)
carbon burning	0.93	$600\,yr$	Na, Ne, Mg
neon burning	1.70	$1\,yr$	Mg, Si
oxygen burning	2.30	$6\,m$	S, P, Si, Mg
silicon burning	4.10	$1\,d$	iron peak elements

Elements heavier than iron:

Since iron has the highest nuclear binding energy, elements heavier than iron can only be produced with energy input. Three processes exist:

s-process: slow capture of n relative to β-decay

r-process: rapid capture of n relative to β-decay

p-process: $(\gamma,$ n) photodisintegration

The r- and s-processes yield neutron-rich elements, the p-process proton-rich elements. The p-process is not as well understood as the r- and s-processes.

s-process contributions come mostly from giants, r-process elements from SN explosions.

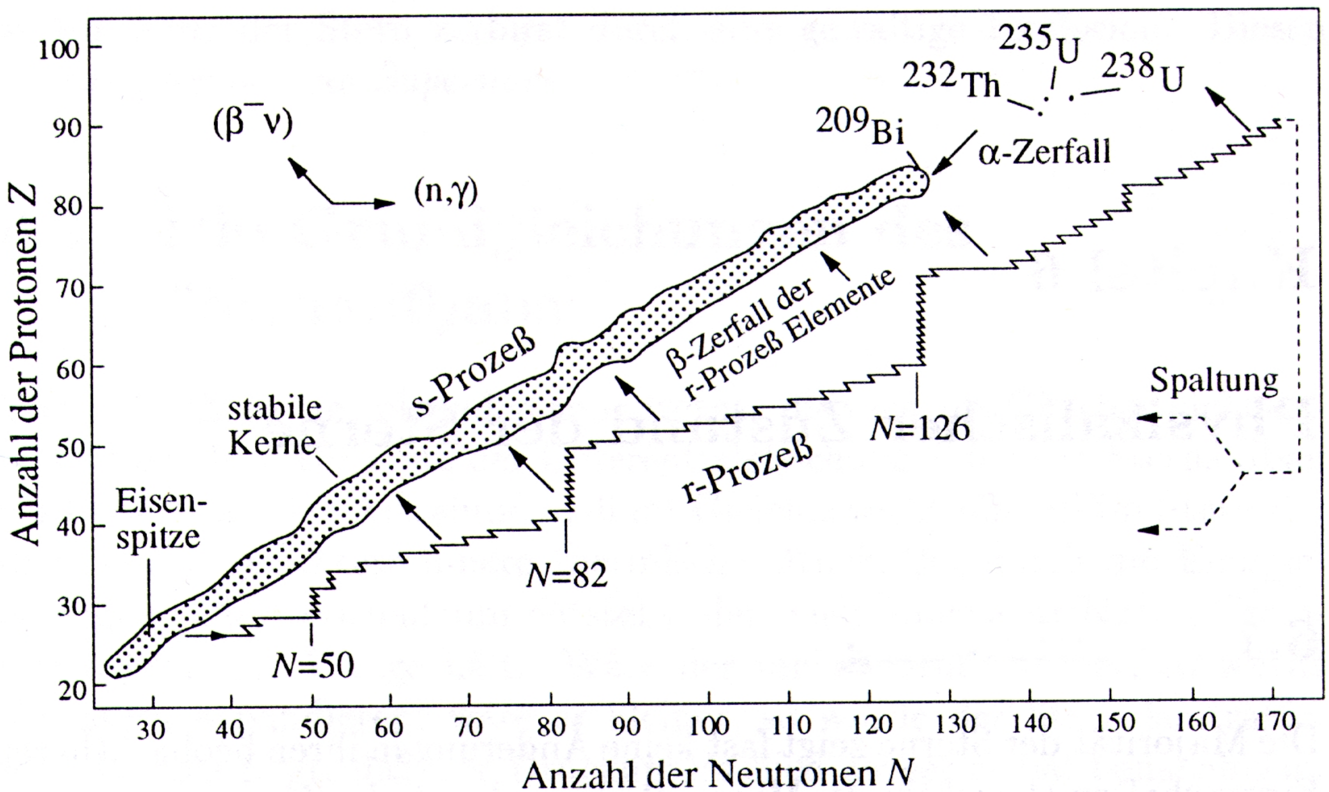


Abbildung 5.11 Der Weg des s– und r–Prozesses in der N–Z–Ebene

see: K. Oberhummer: Kerne und Sterne, J.H.Barth, Leipzig 1993

Place of Production	Elements	Mechanism of Emission into the ISM
CNO-shell-burning of H in *red giants*: $1M_\odot < M_* < 2M_\odot$	^{14}N, He s-process-elements timescale $\geq 10^9 yrs$	convective transport to the star surface, injection into the *wind*, ^{14}N secondary, not in the first generation
incomplete He-burning ($3\alpha \to$ C) in *AGB stars*: $2M_\odot \leq M_* \leq (3-4)M_\odot$	^{12}C, O, He, s-process, little N timescale $\sim 10^7 yrs$	during thermal pulses in AGB phase, mixing and outward transport of C, emission (C-stars, LBVs, Miras $\to$ *PN*)
SN II burning of heavier elements $10M_\odot \leq M_* \leq 40M_\odot$	O, Ne, Mg, Si, S, Ca, r-process, little Fe ($\sim 0.1M_\odot$) timescale $\sim 10^7 yrs$	C-, O-, Ne-, Si-burning in shells, He, C, Ne, N partially by wind, majority by *SN II* -explosion
SN Ia CO-white dwarfs in binaries accrete $10^{-7} M_\odot/yr$ from giant; or: merging of two white dwarfs (less likely)	mostly Fe-group ($\leq 1M_\odot$) little Mg, Si, S, Ar, Ca timescale $\simeq 10^8 yrs$	transformation of accreted H, He into C, O by H-, He-flashes having reached the Chandrasekhar limit ($1.4M_\odot$): initiation of C-burning in core $\to$ outward expansion of subsonic burn-zone $\to$ destruction of WD by SN-explosion

SN Ia Element Production:

Table 7.6. *Element production from SN Ia*

Species	Mass/$M_\odot$	$[X_i/X_{56}]^a$
^{24}Mg	.09	−1.1
^{28}Si	.16	−0.3
^{32}S	.08	−0.4
^{36}Ar	.02	−0.3
^{40}Ca	.04	0.1
^{54}Fe	.14	0.6
^{56}Fe	.61	0.0
^{58}Ni	.06	0.4
Cr–Ni	.86	

[a] Logarithmic element:^{56}Fe ratio relative to solar.
Source: Nomoto *et al.* (1984), model W7; cf. also Thielemann *et al.* (1986).

Timescale for SNIa: about $10^8 yrs$ after formation

see: Pagel (1993) *IAC Winter School*
see: Matteucci (1991) *ASP Conf. 20*, 539

SN II Element Production:

Table 7.2. *Primary element production from massive stars with modest mass loss*

M_{init}	M_{fin} [a]	M_{α} [b]	M_{CO} [c]	He	C	O	Z
120	81	81	59	9.8	0.88	35	42
85	62	62	38	8.1	0.72	23	27
60	47	28	25	6.0	0.70	14	17
40	38	17	14	4.2	0.55	6.8	10
25	25	9	7	3.5	0.40	2.4	4.4
20	19	7	5	2.1	0.30	1.3	2.9
15	15	5	3	1.6	0.20	0.46	1.5
12	12	4	2	1.4	0.10	0.15	0.8
9	9	3	2	1.0	0.06	0.004	0.3
5	5	1	1	0.45			
3	3			0.09			

[a] Final mass at end of carbon burning (or helium burning for lower masses).
[b] Mass of He core at end of carbon or helium burning.
[c] Mass of CO core at end of carbon or helium burning.
Source: Maeder (1992) for the case $Z = 0.001$, $Y = 0.24$.

see: Pagel (1993) *IAC Winter School*
see: Matteucci (1991) *ASP Conf. 20*, 539

Metallicity Dependence of Winds and SNII Explosions:

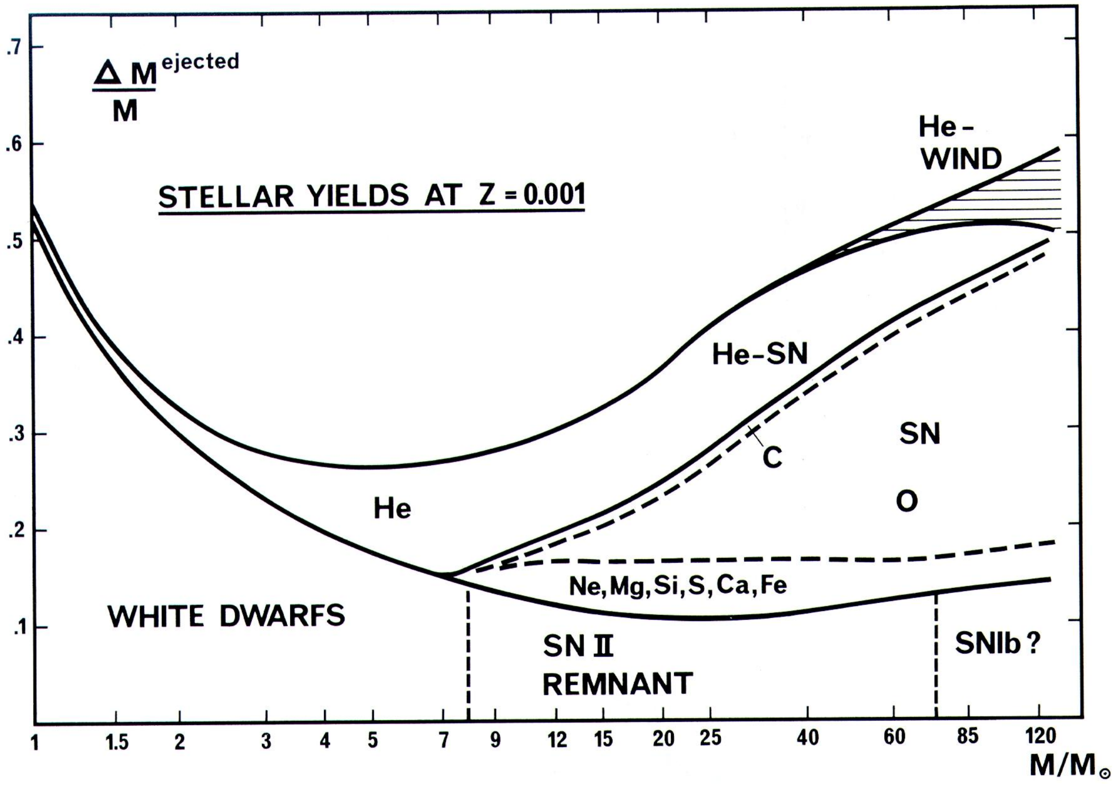


Fig. 5. Stellar yields or mass fractions ejected as a function of the initial masses for metallicity $Z = 0.001$. The quantities shown are the mass fractions p_{im}. The wind contribution is indicated by hatched areas. Some indications on the composition of the ejecta are given. The lower part represents the mass fraction in the remnants

see: Maeder (1992) *A&A*, **264**, 105

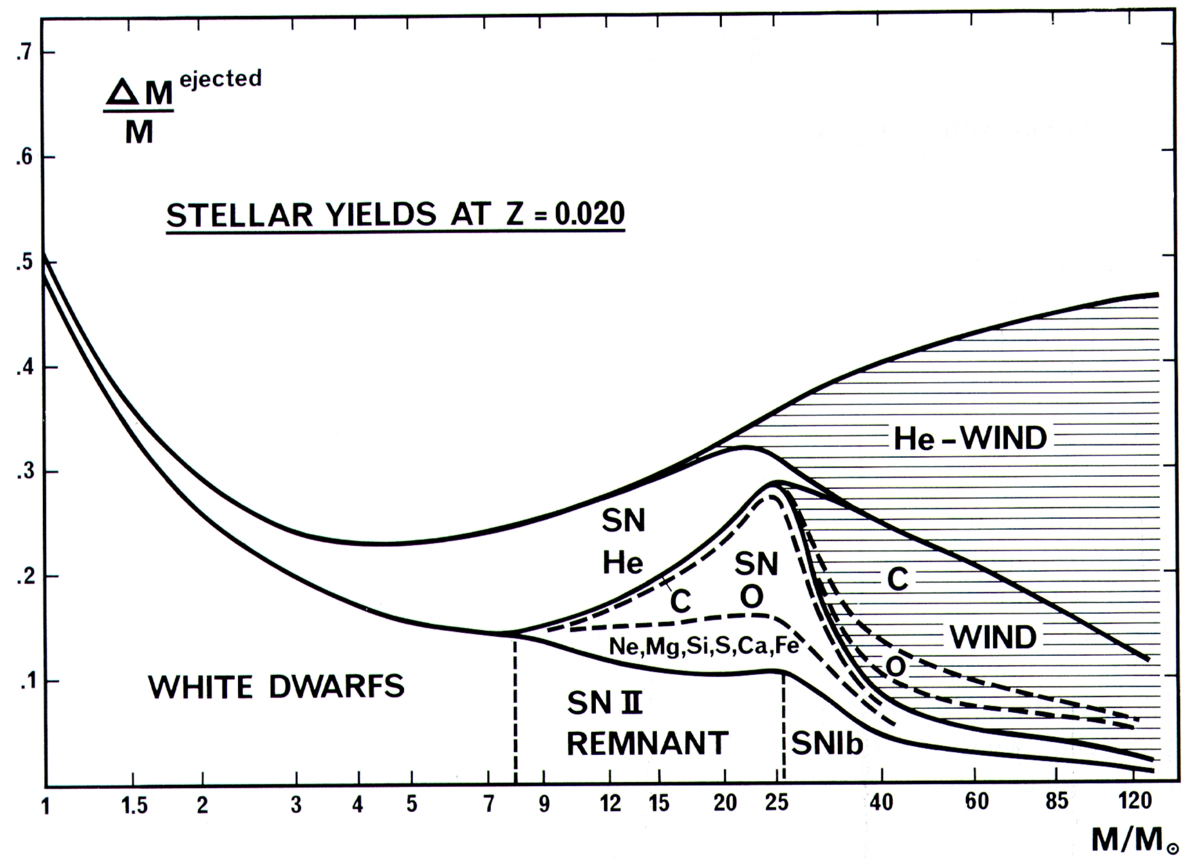


Fig. 6. Same as Fig. 5 for $Z = 0.020$

see: Maeder (1992) *A&A*, **264**, 105

Table 5. Stellar yields for models with $Z = 0.001$ (in solar mass units)

Mass	He	C	O	Z	He	C	O	Z
	Final Ejecta (SN, PN)				Wind and Final Ejecta			
120	2.28	0.89	35.30	41.6	9.81	0.88	35.3	41.6
85	5.56	0.72	22.60	26.7	8.09	0.72	22.6	26.7
60	5.23	0.70	14.20	17.1	5.96	0.70	14.2	17.1
40	4.24	0.55	6.80	9.71	4.24	0.55	6.80	9.71
25	3.52	0.403	2.40	4.45	3.52	0.403	2.40	4.45
20	2.11	0.298	1.27	2.93	2.11	0.298	1.27	2.93
15	1.65	0.196	0.46	1.53	1.65	0.196	0.46	1.53
12	1.35	0.101	0.15	0.83	1.35	0.101	0.15	0.83
9	0.98	0.056	0.004	0.27	0.98	0.056	0.004	0.27
7	0.788				0.788			
5	0.452				0.452			
4	0.285				0.285			
3	0.091				0.091			
2.5	0.065				0.065			
2	0.059				0.059			
1.7	0.026				0.026			
1.5	0.022				0.022			
1.25	0.016				0.017			
1	0.010				0.010			

important:
element-production depends on initial metallicity

see: Maeder (1992) *A&A*, **264**, 105

Table 6. Stellar yields for models with $Z = 0.020$ (in solar mass units)

Mass	He	C	O	Z	He	C	O	Z
	Final Ejecta (SN, PN)				Wind and Final Ejecta			
120	-0.128	0.290	0.18	0.72	42.74	8.04	0.05	10.11
85	-0.392	0.350	0.59	1.56	16.66	13.48	3.96	19.31
60	-0.263	0.333	0.40	1.16	13.52	7.22	1.43	9.85
40	-0.410	0.369	0.62	1.61	6.10	4.88	2.08	8.01
25	0.600	0.319	2.60	4.48	1.54	0.297	2.57	4.48
20	1.520	0.221	1.27	2.73	1.60	0.219	1.27	2.73
15	1.338	0.141	0.41	1.32	1.386	0.140	0.41	1.32
12	1.181	0.072	0.11	0.686	1.195	0.071	0.11	0.686
9	0.871	0.028	0.00	0.173	0.879	0.027	0.00	0.173
7	0.684				0.688			
5	0.401				0.403			
4	0.184				0.185			
3	0.072				0.074			
2.5	0.079				0.080			
2	0.066				0.067			
1.7	0.019				0.020			
1.5	0.015				0.016			
1.25	0.012				0.013			
1	0.010				0.012			
Case of lower $\dot{M}$								
120	-1.304	0.410	2.80	4.56	18.347	20.206	11.18	34.53
85	-1.654	0.393	3.65	5.60	5.243	13.638	13.56	29.92
60	-1.344	0.462	2.95	4.71	6.681	7.741	6.29	15.99
40	-1.406	0.483	3.15	4.91	3.116	3.423	5.17	10.20
25	1.330	0.266	2.55	4.50	1.666	0.259	2.54	4.50
20	1.560	0.233	1.27	2.82	1.585	0.232	1.27	2.82

3.2 Initial Mass Function (IMF)

IMF = distribution of stellar masses at their time of birth. Definition:

$\Phi(m) =$ number of stars formed per mass interval $[m, m + dm]$ and per total mass of formed stars (Unit = $1/mass^2$)

$m\,\Phi(m) =$ mass of stars per mass interval and per total mass
$m\,\Phi(m)dm =$ mass of stars within mass interval and per total mass

resulting normalization:

$$\boxed{\int m\,\Phi(m)dm = 1}$$

Observational determination of IMF:

- from the *solar neighborhood* using star counts and taking into account the lifetime of stars; requires assumptions concerning the star formation history in the solar neighborhood: low-mass stars ($M < 1M_\odot$) have not disappeared since their creation, while O- and B-stars disappear within 10^7 yrs
- from star counts in star forming regions with *infrared photometry* (dust extinction smaller in IR)

Note: it is unclear if the IMF is *bimodal*, i.e. whether high-mass and low-mass stars form in separate regions, or, whether they form in the same region at different times.

Common analytical approximations:

Salpeter-IMF: (*ApJ* **121**, 161 (1955))

$0.1 \leq \frac{m}{M_\odot} \leq 100$: $\Phi(m) \simeq 0.17 m^{-2.35}$

Miller-Scalo-IMF: (see also figure)

	$\Phi(m) =$	mass-fraction
$0.1 \leq \frac{m}{M_\odot} \leq 0.5$	$0.93 m^{-0.85}$	0.31
$0.5 \leq \frac{m}{M_\odot} \leq 1.0$	$0.46 m^{-1.85}$	0.31
$1.0 \leq \frac{m}{M_\odot} \leq 3.16$	$0.46 m^{-3.4}$	0.26
$3.16 \leq \frac{m}{M_\odot} \leq 100$	$0.21 m^{-0.27}$	0.12

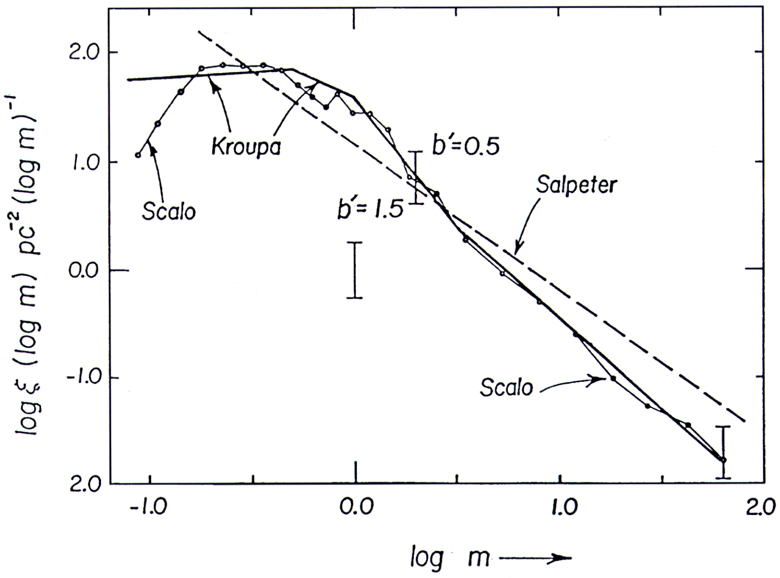


Fig. 7.1. Local IMF after Scalo (1986) with $b' = 1$ (points joined by thin lines), Kroupa *et al.* (1991) including 'distance effect' (thick lines below $m = 1$) and two additional power-law segments approximating Scalo's IMF. Salpeter's law is shown by a broken line. The IMFs are normalized to a total mass of stars ever born of $37 M_\odot \mathrm{pc}^{-2}$ between mass limits $m = 0.1$ and $m = 100$.

see: Pagel: Nucleosynthesis and Chemical Evolution of Galaxies, Cambridge University Press 1997

3.3 Star Formation Rate

$\Psi =$ total mass of stars formed per unit time and unit volume

for solar neighborhood: $\Psi'_\odot \simeq 3 \frac{M_\odot}{pc^2\, Gyrs}$

$\Rightarrow \; \Psi(t) \cdot \Phi(m) \; =$ number of stars formed per unit mass, per unit time and per unit volume

3.4 Basic Chemical Evolution Model

see: Tinsley 1980 and corrections by Maeder 1992

- conservation laws:

(1) $M = M_s + M_g$
$$\begin{cases} M & = \text{total mass} \\ M_s & = \text{mass in stars} \\ M_g & = \text{mass in gas} \end{cases}$$

(2) $\frac{dM}{dt} = f - e$
$$\begin{cases} f & = \text{rate of infalling gas} \\ e & = \text{rate of ejected gas} \end{cases}$$

(3) $\frac{dM_s}{dt} = \Psi - E$
$$\begin{cases} \Psi & = \text{star formation rate} \\ E & = \text{gas ejection rate of all stars} \end{cases}$$

(4) $\frac{dM_g}{dt} = -\Psi + E + f - e$

- gas ejection rate of all stars:

$$\text{(5)}\quad E(t) = \int_{m_t}^{\infty} [m - w_m] \Psi_{t-\tau(m)} \Phi(m) dm$$

m_t:	turnoff mass at time t = lowest mass of stars dying at time t
$m - w_m$:	ejected mass
$\Psi_{t-\tau(m)}\Phi(m)$:	birth rate at $t - \tau(m)$ = death rate at time t, stars of different generations are involved
$\tau(m)$:	main-sequence lifetime at mass m

Remnant mass
$$\begin{cases} w_m = 0.11m + 0.45M_\odot & (m < 6.8M_\odot) \\ w_m = 1.5M_\odot & (m \geq 6.8M_\odot) \end{cases}$$

N.B.: only assumption: mass loss at the end in negligible time
(wrong only for $\Delta t < 10^6$ because of O-star winds)

- evolution of the abundance Z (relative mass fraction) for one or more metals:

$$\text{(6)} \quad \frac{d(ZM_g)}{dt} = -Z\Psi + E_Z + Z_f \cdot f - Ze$$

E_Z: ejection rate of metals from stars, SNe etc.
$Z_f \cdot f$: infalling metals per time
ZM_g: mass of metals in the gas

- ejection rate of metals

$$\text{(7)} \quad E_Z(t) = \int_{m_t}^{\infty} [(m - w_m)Z_{t-\tau(m)} + mp_{Zm}]\Psi_{t-\tau(m)}\Phi(m)dm$$

$(m - w_m)Z_{t-\tau(m)}$: mass of metals that at time $t - \tau(m)$ were locked in a star of mass m and are now ejected with the envelope at time t; often also written as $(m - w_m - mp_{Zm})Z_{t-\tau(m)}$ (see Tinsley 1980)

mp_{Zm}: new metals produced by a star of mass m which originally formed from gas with metallicity Z
Note: for some elements (e.g. Li) $p_{Zm} < 0$

Important assumption in equation (6) and (7):

Instantaneous mixing of produced metals with ISM

- returned mass per mass of stars formed

$$(8) \quad R = \int_{m_1}^{\infty} (m - w_m)\Phi(m)dm$$

equation (8) is independent of Ψ and is valid for one (!) generation of stars.

Example:

Let us assume that stars with total mass M were formed 10^{10} yrs ago.
Then, today $m_1 = M_\odot$, the mass of the remnants is $(1 - R)M$ and
the returned gas mass is RM.

- Yield (mass of produced metals per remnant mass)

$$(9) \quad y = \frac{1}{1-R} \int_{m_1}^{\infty} m p_{Zm} \Phi(m) dm$$

yield y: mass of newly produced metals by a populations of stars which after their deaths have a sum of remnant masses of $1 M_\odot$.

3.5 Solution of the Chemical Evolution Equations

- Evaluation by numerical methods: Different timescales for winds, PN, SNII and SNI lead to time-dependent element abundances. In the beginning enrichment by winds and SNII, after $\sim 10^8$ yrs enrichment also by SNI.
  $\Rightarrow$ Oldest stars in the galaxy have lower Fe-fraction (SNI) than younger stars and the sun.

- For simplified approximations the **Instantaneous Recycling Approximation** (I.R.A.) is used, i.e. it is assumed that the newly produced elements will be instantaneously returned to and uniformly mixed with the ISM.
  This assumption is only valid, if the SFR for light elements (O, C, N, Mg, . . . created by SNII-explosions) is nearly constant over a timescale of 10^7 yrs.
  As far as heavier elements (Fe, created by SNI) are concerned, the SFR must not change on timescales of $10^{8..9}$ yrs.

Element Abundances in Solar Neighborhood:

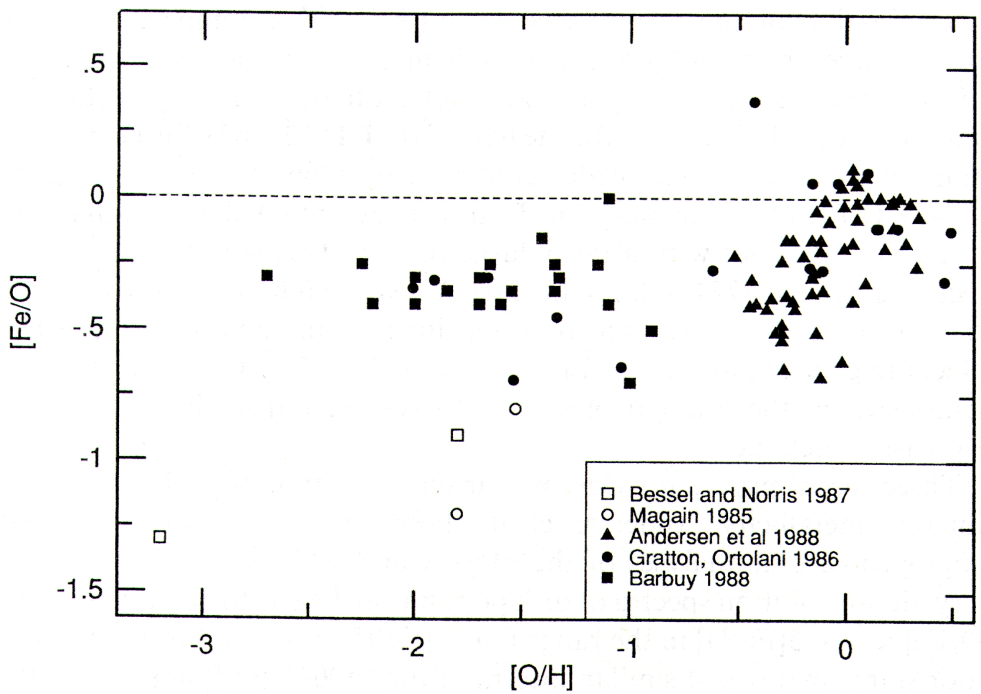


Figure 2 A comparison of O and Fe abundances in field stars, with O taken as the independent metallicity indicator.

[Fe/O]= logarithmic ratio between mass of Fe and mass of O normalized to sun:

$$= \log \frac{\rho_{Fe}/\rho_{Fe_\odot}}{\rho_O/\rho_{O_\odot}}$$

see: Wheeler et al. (1989) *ARAA* **27**

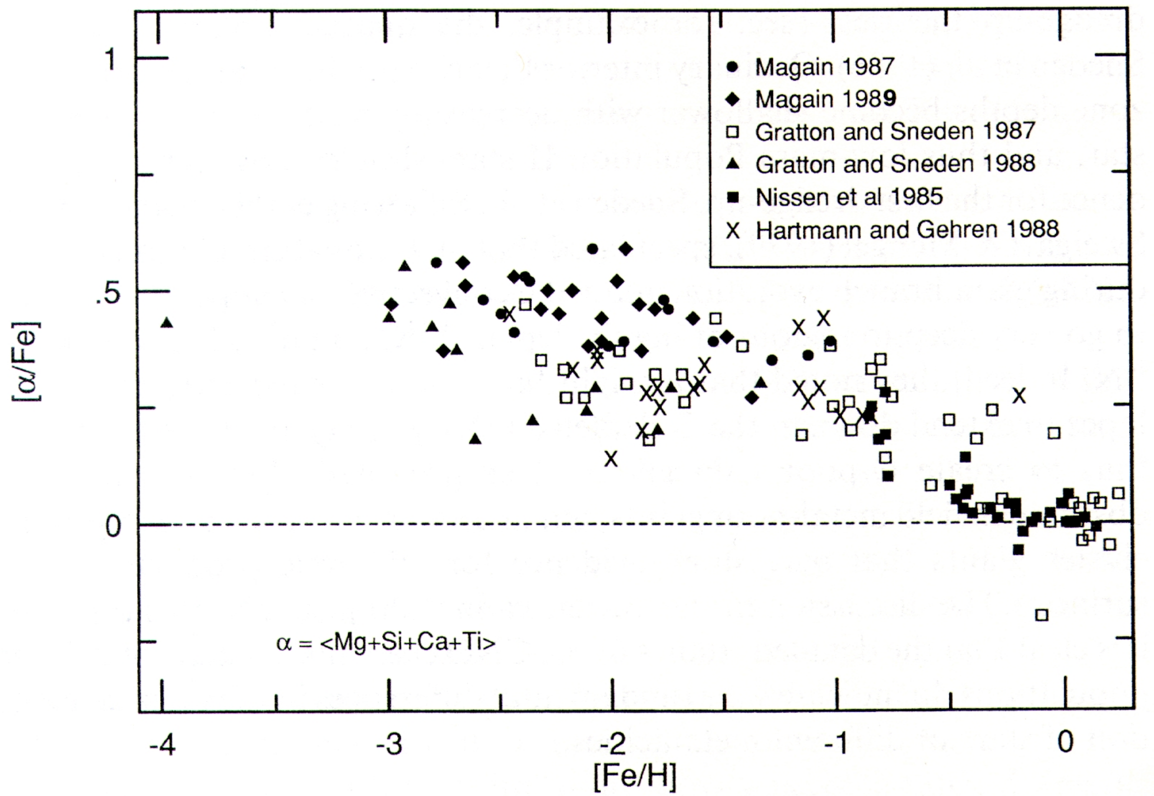


Figure 3 The average α-element abundances as functions of the traditional metallicity indicator Fe. Note that the averaging of the four α-element abundances may blur the possibly real variations with Fe among this group.

see: Wheeler et al. (1989) *ARAA* **27**

Element Abundances of the Sun: Typical for Galactic Disk Stars:

Tabelle 4.9.1. *Elementhäufigkeiten* log N im Sonnensystem: *Sonne* (⊙) nach H. Holweger (1985) und *kohlige Chondrite vom Typ* C1 nach E. Anders und M. Ebihara (1982). Normierung auf Wasserstoff log N(H) = 12.0, Anpassung der solaren und meteoritischen Häufigkeitsverteilungen bei Silizium log N(Si) = 7.6. Bestimmung der Sonnenhäufigkeiten aus der Photosphäre mit Ausnahme von He, Ne, Ar (Korona bzw. Protuberanzen) und Tl (Sonnenflecken). Meteorite: C1-Chondrite bis auf Be, B, Br, Rh, I, für die andere Chondrite herangezogen wurden. Für Kr, Xe, Hg geschätzte Werte aus Interpolationen. Radioaktive Elemente: Th, U Angabe der *heutigen* Häufigkeiten; bei der Entstehung des Sonnensystems vor $4.5 \cdot 10^9$ a (2.8.24) waren die Häufigkeiten um $\delta \log N = 0.2$ (Th) bzw. 0.3 (U) höher

	⊙	C1		⊙	C1		⊙	C1		⊙	C1
1 H	12.0	–	22 Ti	5.1	5.0	44 Ru	1.8	1.9	66 Dy	1.1	1.2
2 He	*11.0*	–	23 V	4.1	4.1	45 Rh	1.1	*1.1*	67 Ho	0.3	0.6
3 Li	1.1	3.4	24 Cr	5.8	5.7	46 Pd	1.7	1.7	68 Er	0.9	1.0
4 Be	1.2	*1.5*	25 Mn	5.4	5.6	47 Ag	0.9	1.3	69 Tm	0.3	0.1
5 B	2.5	*3.0*	26 Fe	7.6	7.6	48 Cd	1.9	1.8	70 Yb	1.1	1.0
6 C	8.6	–	27 Co	4.9	5.0	49 In	1.7	0.9	71 Lu	0.8	0.2
7 N	8.0	–	28 Ni	6.2	6.3	50 Sn	1.9	2.2	72 Hf	0.9	0.8
8 O	8.9	–	29 Cu	4.2	4.3	51 Sb	1.0	1.1	73 Ta	–	0.0
9 F	4.6	4.5	30 Zn	4.6	4.7	52 Te	–	2.3	74 W	1.1	0.7
10 Ne	*7.6*	–	31 Ga	2.9	3.2	53 I	–	*1.6*	75 Re	–	0.3
11 Na	6.3	6.4	32 Ge	3.5	3.7	54 Xe	–	(2.2)	76 Os	1.4	1.5
12 Mg	7.5	7.6	33 As	–	2.4	55 Cs	–	1.2	77 Ir	1.4	1.4
13 Al	6.4	6.5	34 Se	–	3.4	56 Ba	2.1	2.2	78 Pt	1.8	1.7
14 Si	7.6	7.6	35 Br	–	*2.7*	57 La	1.1	1.3	79 Au	1.1	0.9
15 P	5.4	5.6	36 Kr	–	(3.3)	58 Ce	1.6	1.7	80 Hg	–	(1.3)
16 S	7.2	7.3	37 Rb	2.6	2.5	59 Pr	0.7	0.8	81 Tl	*0.9*	0.9
17 Cl	–	5.3	38 Sr	3.0	3.0	60 Nd	1.4	1.5	82 Pb	1.9	2.1
18 Ar	*6.7*	–	39 Y	2.2	2.3	62 Sm	0.8	1.0	83 Bi	–	0.8
19 K	5.1	5.2	40 Zr	2.6	2.6	63 Eu	0.5	0.6	90 Th	0.2	0.1
20 Ca	6.4	6.4	41 Nb	1.4	*1.5*	64 Gd	1.1	1.1	92 U	–	−0.4
21 Sc	3.1	3.1	42 Mo	1.9	2.0	65 Tb	0.2	0.4			

see: Unsöld/Baschek

3.5.1 Instantaneous Recycling Approximation

$\Psi(t-\tau_m) \simeq \Psi(t)$ and homogeneous mixing of the gas.

using R and y:

$$(5) \rightarrow (5') \ : \ E(t) = R\Psi(t)^{1}$$
$$(7) \rightarrow (7') \ : \ E_Z(t) = RZ(t)\Psi(t) + (1-R)y(t)\Psi(t)^{1}$$

inserting $(7')$ in (6):

$$\frac{d(ZM_g)}{dt} = -Z\Psi + RZ(t)\Psi(t) + (1-R)y(t)\Psi(t) + Z_f \cdot f - Ze$$

$$(6) \rightarrow (6') \ : \ \frac{d(ZM_g)}{dt} = (1-R)(-Z+y)\Psi + Z_f \cdot f - Ze$$

[1]if IMF timedependent: $R = R(t)$

inserting $(5')$ in (3):

$$(A) \quad \frac{dM_s}{dt} = (1-R)\Psi(t)$$

inserting $(5')$ in (4):

$$(B) \quad \frac{dM_g}{dt} = -(1-R)\Psi(t) + f - e$$

inserting (B) in $(6')$:

$$(C) \quad M_g \frac{dZ}{dt} = (1-R)y(t)\Psi(t) + (Z_f - Z)f + eZ$$

(A)**,** (B)**,** (C) **are the equations governing chemical evolution in the I.R.A.**

3.5.2 Closed-Box Model

Closed-box model: simple example of chemical evolution

$\rightarrow$ Assume a closed box containing only gas with metallicity $Z = 0$ and no stars.

Problem: Calculate the metallicity of the gas and the stars at any time after the onset of star formation (assuming I.R.A.).

With $f = e = 0$, $M_g(t = 0) = M$, $M_s(t = 0) = 0$ (= closed-box model):

$(B) \quad \rightarrow \frac{dM_g}{dt} = -(1 - R)\Psi(t)$

$(C) \quad \rightarrow M_g \frac{dZ}{dt} = (1 - R)y(t)\Psi(t)$

and (B)/(C) yields:

$$\frac{1}{M_g}\frac{dM_g}{dZ} = -\frac{1}{y}$$

$$\ln M_g \Big|_{M}^{M_g(t)} = \int_0^{Z(t)} -\frac{dZ}{y} \simeq -\frac{Z}{\bar{y}}$$

(for some elements the '$\simeq$' this is a strong simplification)

Thus the metallicity of the gas becomes:

$$\boxed{Z(t) = \bar{y} \ln \frac{M_g(t=0)}{M_g(t)}} \tag{3.1}$$

Important:

- The above formula is only valid for $Z \ll 1$ or $\frac{M_g(t)}{M} > 0$.

  For $M_g(t) \to 0$, $y \to 0$ and $\int_0^Z -\frac{dZ}{y}$ cannot be approximated by $-\frac{Z}{\bar{y}}$.

- $Z(t)$ depends only on $\frac{M_g(t)}{M}$ and thus not explicitely on t. If $\frac{M_g}{M}$ is known, Z can be determined.

Example:

For $Z < 0.05$ and Salpeter IMF: $\bar{y} \simeq 0.01...0.02$ (by chance $Z_\odot \simeq 0.02$) solar metallicity is reached for:

$$Z_\odot = Z_\odot \ln \frac{M}{M_g(t)} \quad \rightarrow \quad M_g(t) = 0.37M$$

For a simple estimate one can use:

$$\boxed{\frac{Z}{Z_\odot} \simeq \ln \frac{M}{M_g(t)}}$$

What is the metallicity of the stars?

Assuming a closed-box model, the stars and the gas together must contain all elements ever produced. Therefore:

$$\underbrace{Z_s M_s}_{1} + \underbrace{Z M_g}_{2} = \underbrace{\int_0^t \int_0^\infty m p_{Zm} \Psi(t')\, \Phi(m)\, dm\, dt'}_{3} = \int_0^t (1-R) y \Psi\, dt \simeq (1-R) \underbrace{\bar{y}\, \bar{\Psi}}_{4}\, t$$

1: average metallicity of stars (without metals locked in remnants)
2: metallicity of gas
3: mass of all metals ejected into ISM
4: averaged values with assumption: $\overline{y\Psi} \simeq \bar{y}\, \bar{\Psi}$

Therefore:

$$Z_s M_s + Z M_g = (1-R)\, \bar{y}\, \bar{\Psi}\, t$$

Integrating (A) $\frac{dM_s}{dt} = (1-R)\Psi(t)$ leads to

$$M_s = (1-R)\,\bar{\Psi}\,t$$

Combined with former equation yields:

$$\boxed{Z_s M_s \simeq \bar{y}\,M_s - Z M_g}$$ (Closed-box model)

Therefore at the end, when $M_g \ll M_s$:

$$\boxed{M_g \ll M_s \Longrightarrow Z_s \simeq \bar{y}}$$

i.e., the average metallicity of the stars cannot be larger than the average yield!